On Some Hamiltonian Structures of Painlevé Systems, I

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§0. Introduction

In this series of papers, we study some Hamiltonian structures of Painlevé systems (H_J), $J = VI, V, IV, III, II, I$, namely, symplectic structures of the spaces for Painlevé systems constructed by K. Okamoto([7]), and a characterization of Painlevé systems by their spaces.

As is well known, P. Painlevé and B. Gambier discovered, at the beginning of this century, six nonlinear second order differential equations free from movable branch points, which are now called the Painlevé equations. We denote them by P_J, $J = VI, V, IV, III, II, I$. For example, the sixth Painlevé equation P_{VI} is given by

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{x-t}\right)\frac{dx}{dt} \\ &\quad + \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{x^2} + \gamma\frac{t-1}{(x-1)^2} + (\frac{1}{2}-\delta)\frac{t(t-1)}{(x-t)^2}\right],\end{aligned}$$

where x and t are complex variables, and α, β, γ and δ are complex constants. The most important property of the Painlevé equations is the so called *Painlevé property*, namely, *every solution of each Painlevé equation has neither movable branch points nor movable essential singularities.* Let $\Xi_J \subset \mathbf{P} = \mathbf{C} \cup \{\infty\}$ be the set of the fixed singular points of P_J and let $B_J = \mathbf{P} - \Xi_J$. Then the Painlevé property is stated as: *any local solution $x(t)$ of P_J (determined by an arbitrary initial condition $x(t_0) = x_0$, $(dx/dt)(t_0) = x_1$ with $t_0 \in B_J$ and with a certain condition on x_0, for example, with $x_0 \neq 0, 1, t_0$ for $J = VI$) can be meromorphically continued along any curve in B_J.*

We know that each Painlevé equation P_J is equivalent to a Hamiltonian system

$$(H_J) \qquad dx/dt = \partial H_J/\partial y, \qquad dy/dt = -\partial H_J/\partial x,$$

where H_J is a polynomial of x and y of which the coefficients are rational functions of t holomorphic in B_J ([4],[8]). For example, H_{VI} is given by

$$\begin{aligned}H_{VI}(x,y,t) =& \frac{1}{t(t-1)}[x(x-1)(x-t)y^2 - \{\kappa_0(x-1)(x-t) \\ &+ \kappa_1 x(x-t) + (\kappa_t - 1)x(x-1)\}y + \kappa(x-t)],\end{aligned} \tag{0.1}$$

where

$$\kappa = \frac{1}{4}[(\kappa_0 + \kappa_1 + \kappa_t - 1)^2 - {\kappa_\infty}^2], \tag{0.2}$$

$\kappa_0,\ \kappa_1,\ \kappa_t$ and κ_∞ being complex constants. Here the equivalence of P_J and (H_J) means that if we eliminate the variable y in the system (H_J) then we obtain the equation P_J in x, provided the constants in P_J and (H_J) are related by certain relations, for example, by

$$\alpha = \frac{1}{2}{\kappa_\infty}^2, \quad \beta = \frac{1}{2}{\kappa_0}^2, \quad \gamma = \frac{1}{2}{\kappa_1}^2, \quad \delta = \frac{1}{2}{\kappa_t}^2, \tag{0.3}$$

for $J = VI$. We call the Hamiltonian system (H_J) the *J-th Painlevé system.* In order that the elimination of y is possible, it is necessary and sufficient that $x(t)$ is not a constant which can not be a solution of P_J. However, in the case where $x(t)$ is such a constant, $y(t)$ is a solution of a certain Riccati equation. Therefore, *if* $(x(t), y(t))$ *is a solution of* (H_J) *determined by an arbitrary initial condition* $x(t_0) = x_0 \in \mathbf{C},\ y(t_0) = y_0 \in \mathbf{C}$ *with* $t_0 \in B_J$ *then both* $x(t)$ *and* $y(t)$ *can be meromorphically continued along any curve in* B_J *with a starting point* t_0. We also call the property the *Painlevé property for* (H_J).

Let $\mathcal{Q}_J = (\mathbf{C}^2 \times B_J, \pi_J, B_J)$ be a trivial fiber space over B_J. Then the Painlevé system (H_J) determines in $\mathbf{C}^2 \times B_J$ a complex 1-dimensional nonsingular foliation such that every leaf passing through a point in $\mathbf{C}^2 \times t$ with $t \in B_J$ is transversal to the fiber $\mathbf{C}^2 \times t$, because the Hamiltonian function H_J is a polynomial of x and y of which the coefficients are holomorphic in B_J. However, for a solution $(x(t), y(t))$ of (H_J), the function $x(t)$ or $y(t)$ may have movable poles in B_J in general. Therefore, the foliation has *not* the following property: *for any point* $(x_0, y_0, t_0) \in \mathbf{C}^2 \times B_J$ *and any curve* l *starting from* t_0, *the curve* l *can be lifted in a leaf through the point* (x_0, y_0, t_0). A nonsingular foliation of which every leaf is transversal to the fibers and moreover with the above property is said to be *uniform.*

We now cite a work by K. Okamoto([7]) which is directly related to the study in this series of papers. He constructed a minimal space in which every solution of (H_J) stays, more precisely, a fiber space $\mathcal{P}_J = (E_J, \pi_J, B_J)$ over B_J such that

(i) $\mathcal{P}_J$ contains $\mathcal{Q}_J$ as a fiber subspace,

(ii) the system (H_J) of differential equations defined in the total space $\mathbf{C}^2 \times B_J$ of $\mathcal{Q}_J$ is holomorphically extended in that E_J of $\mathcal{P}_J$ and it determines a uniform foliation in E_J, namely, for any point $P_0 \in E_J$ ($\pi_J(P_0) = t_0 \in B_J$) and any curve l in B_J with a starting point t_0, the solution $P = P(t)$ ($\pi_J(P(t)) = t$) of the extended system satisfying $P(t_0) = P_0$ is holomorphically continued in E_J over the curve l.

(iii) every leaf in the total space E_J intersects with the total space $\mathbf{C}^2 \times B_J$ of $\mathcal{Q}_J$.

Every fiber $E_J(t) = {\pi_J}^{-1}(t),\ t \in B_J$ is called *a space of initial conditions of* (H_J), because there exists a bijection from $E_J(t)$ to the set of all the solutions of (H_J). We call the total space E_J *the space for* (H_J) in our papers.

The fiber space $\mathcal{P}_J$ is constructed as follows. Firstly, we take a minimal compactification $\overline{\Sigma}_\epsilon$ of $\mathbf{C}^2$, which depends on the values of constants in (H_J). Secondly, we apply a finite number of quadratic transformations to the space $\overline{\Sigma}_\epsilon \times t$ for every $t \in B_J$ by carefully observing the forms of Pfaffian systems in new variables transformed from the original system (H_J), and obtain a compact space $\overline{E_J(t)}$. Then we define a fiber space $(\overline{E}_J, \pi_J, B_J)$ by $\overline{E}_J = \cup_{t \in B_J} \overline{E_J(t)} \times t$. Lastly, we remove, from each fiber $\overline{E_J(t)}$, a finite number of divisors which consist of vertical leaves and singular points of the foliation, and obtain $E_J(t),\ t \in B_J$. (Here a *vertical* leaf is, by definition, a leaf which is completely included in a fiber.) Then the total space E_J of $\mathcal{P}_J$ is defined by $E_J = \cup_{t \in B_J} E_J(t) \times t$. It is proved by the Painlevé property for (H_J) that the fiber space $\mathcal{P}_J$ has the above properties (i),(ii) and (iii). It should be noted that every fiber $E_J(t)$ is noncompact.

The purpose of this series, is to study the space E_J for the Painlevé system (H_J) for each J, namely, (a) to suitably choose a finite number of coordinate neighborhoods (which is an open covering of E) and coordinate systems of E_J so that the transition functions are rational and symplectic, (b) to prove a certain uniqueness of Hamiltonian systems on the space E_J. We remark that, in every coordinate neighborhood of ours, the Hamiltonian function H_J is expressed as a polynomial of the coordinates, and the above (b) implies that a global analysis of the Painlevé system (H_J) reduces to a geometry of the space E_J.

In this paper, we give our results for the sixth Painlevé system (H_{VI}). In Section 1, we sate the main results, Theorems 1 and 2. The proofs of the theorems are given in Sections 2,3, and 4. In order to prove Theorem 1, we have to review the construction of the space E_{VI}, which is done in Section 2. In Section 4, we prove Theorem 2 by solving linear equations for the coefficients of the Taylor expansion of a Hamiltonian function.

§1. Statement of main results

In order to explain our results, we recall the definition of a symplectic transformation and its properties. Let $x = x(X, Y, t),\ y = y(X, Y, t),\ t = t$ be a biholomorphic mapping from a domain in $\mathbf{C}^3 \ni (X, Y, t)$ into $\mathbf{C}^3 \ni (x, y, t)$. We say that the mapping is *symplectic* if

$$dy \wedge dx = dY \wedge dX, \tag{1.1}$$

where t is considered as a constant or a parameter. Suppose that the mapping is symplectic. Then any Hamiltonian system $dx/dt = \partial H/\partial y,\ dy/dt = -\partial H/\partial x$ is transformed to $dX/dt = \partial K/\partial Y,\ dY/dt = -\partial K/\partial X$ where

$$dy \wedge dx - dH \wedge dt = dY \wedge dX - dK \wedge dt. \tag{1.2}$$

Here t is considered as a variable. By the equation (1.2), the function K is determined from H uniquely modulo functions of t, namely, modulo functions independent of X and Y.

Now let

$$\epsilon(\pm) = (\kappa_0 + \kappa_1 + \kappa_t - 1 \pm \kappa_\infty)/2, \tag{1.3}$$

where $\kappa_0,\ \kappa_1,\ \kappa_t$ and κ_∞ are the constants in the Hamiltonian function H_{VI} given by (0.1), then we have

Theorem 1. The total space $E = E_{VI}$ of the fiber space $\mathcal{P}_{VI} = (E_{VI}, \pi_{VI}, B_{VI})$ over $B = B_{VI} = \mathbf{C} - \{0, 1\}$ for the sixth Painlevé system (H_{VI}) is obtained by glueing six copies of $\mathbf{C}^2 \times B$:

$$\begin{aligned}
V(00) \times B = \mathbf{C}^2 \times B &\ni (x, y, t) = (x(00), y(00), t),\\
V(0\infty) \times B = \mathbf{C}^2 \times B &\ni (x(0\infty), y(0\infty), t),\\
V(1\infty) \times B = \mathbf{C}^2 \times B &\ni (x(1\infty), y(1\infty), t),\\
V(t\infty) \times B = \mathbf{C}^2 \times B &\ni (x(t\infty), y(t\infty), t),\\
V(\infty 0+) \times B = \mathbf{C}^2 \times B &\ni (x(\infty 0+), y(\infty 0+), t),\\
V(\infty 0-) \times B = \mathbf{C}^2 \times B &\ni (x(\infty 0-), y(\infty 0-), t),
\end{aligned}$$

via the following symplectic transformations

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty), \tag{1.4}$$

$$x(00) = 1 + y(1\infty)(\kappa_1 - x(1\infty)y(1\infty)), \quad y(00) = 1/y(1\infty), \tag{1.5}$$

$$x(00) = t + y(t\infty)(\kappa_t - x(t\infty)y(t\infty)), \quad y(00) = 1/y(t\infty), \tag{1.6}$$

$$x(00) = 1/x(\infty 0+), \quad y(00) = x(\infty 0+)(\epsilon(+) - x(\infty 0+)y(\infty 0+)), \tag{1.7}$$

$$x(\infty 0+) = y(\infty 0-)(\kappa_\infty - x(\infty 0-)y(\infty 0-)), \quad y(\infty 0+) = 1/y(\infty 0-), \tag{1.8}$$

where $V(00) \times B$ is the original space in which the Hamiltonian function $H_{VI}(x, y, t)$ is defined.

Let us denote by I the set of six labels:

$$I = \{00,\ 0\infty,\ 1\infty,\ t\infty,\ \infty 0+,\ \infty 0-\}. \tag{1.9}$$

We consider each $V(*)\times B$, $* \in I$ as a coordinate neighborhood of E. It is easy to see that a fiber $E(t) = {\pi_{VI}}^{-1}(t),\ t \in B$ is a disjoint union of $V(00) = \mathbf{C}^2$ and five complex lines $\{y(*) = 0\},\ * \neq 00, \infty 0+$ and $\{x(\infty 0+) = 0\}$.

Because every coordinate transformation is symplectic, the Hamiltonian system (H_{VI}) in $V(00) \times B$ is also written as a Hamiltonian system in each $V(*) \times B$, $* \in I$. We denote by $H(*)$ or by $H(*; x(*), y(*), t)$ the Hamiltonian function in $V(*) \times B$ transformed from $H_{VI}(x, y, t)$. By using (1.2), we see that $H(*; x(*), y(*), t)$ is a *polynomial* of $x(*)$ and $y(*)$ of which the coefficients are rational functions of t holomorphic in B. This fact should be remarked. Let us consider our Hamiltonian system, for example, in $V(0\infty) \times B$: $dx(0\infty)/dt = \partial H(0\infty)/\partial y(0\infty)$, $dy(0\infty)/dt = -\partial H(0\infty)/\partial x(0\infty)$. Since $H(0\infty)$ is a polynomial of $x(0\infty)$ and $y(0\infty)$ of which the coefficients are holomorphic in B, our system has a unique local solution $(x(0\infty)(t), y(0\infty)(t))$ satisfying $x(0\infty)(t_0) = h,\ y(0\infty)(t_0) = 0$, for any $h \in \mathbf{C}$ and for an arbitrarily fixed $t_0 \in B$. By means of (1.4), the solution corresponds to a solution $(x(h;t), y(h;t))$ of (H_{VI}) such that $\lim_{t\to t_0} x(h;t) = 0,\ \lim_{t\to t_0} y(h;t) = \infty$. Then there exist infinitely many solutions $\{(x(h;t), y(h;t)) | h \in \mathbf{C}\}$ of (H_{VI})

which pass through the point $(x,y) = (0,\infty)$. The label 0∞ indicates that $(x(0\infty), y(0\infty))$ is a coordinate system which separates infinitely many solutions of (H_{VI}) passing through the point $(x,y) = (0,\infty)$.

It may be noticed that, for any $t_0,\ t_1 \in B$, $E(t_0)$ is isomorphic to $E(t_1)$ not only as complex manifold but also as symplectic manifold. The property is easily shown as follows. Let us take a curve l in B joining t_0 to t_1. For a point $P_0 \in E(t_0)$, we obtain a unique solution $P = P(t)$ $(\pi_{VI}(P(t)) = t)$ of our Hamiltonian system passing through the point P_0. The solution can be holomorphically continued in E over l and it determines a point $P_1 = P(t_1) \in E(t_1)$. The transformation which maps P_0 to P_1 is biholomorphic and symplectic.

We also notice the relations

$$
\begin{aligned}
x(00)y(00) &= \kappa_0 - x(0\infty)y(0\infty),\\
(x(00)-1)y(00) &= \kappa_1 - x(1\infty)y(1\infty),\\
(x(00)-t)y(00) &= \kappa_t - x(t\infty)y(t\infty),\\
x(00)y(00) &= \epsilon(+) - x(\infty0+)y(\infty0+)\\
&= \epsilon(-) - x(\infty0-)y(\infty0-),\\
x(\infty0+)y(\infty0+) &= \kappa_\infty - x(\infty0-)y(\infty0-),
\end{aligned}
\tag{1.10}
$$

which are useful in studying the behavior of a solution $(x(t), y(t))$ of (H_{VI}).

We next consider a problem if there exist Hamiltonian systems defined on $E = E_{VI}$ other than the sixth Painlevé system (H_{VI}). By a Hamiltonian system holomorphic on E, we mean a family of Hamiltonian functions $\{K(*; x(*), y(*), t)\}_{*\in I}$ such that each $K(*) = K(*; x(*), y(*), t)$ is holomorphic in $V(*) \times B$ and every $K(*)$ is the transform of $K(00)$ by the symplectic transformation between $(x(*), y(*), t)$ and $(x(00), y(00), t)$. We remark that a Hamiltonian system $\{K(*)\}_*$ on E does not define a function on E but the difference $\{K(*) - K'(*)\}_*$ of any two Hamiltonian systems $\{K(*)\}_*$ and $\{K'(*)\}_*$ on E defines a function on E, by adding functions of t if it is necessary. Let $\{K(*)\}_*$ be a holomorphic Hamiltonian system on E. We say that it is meromorphically extendable to the space $\overline{E}$ if each $K(*)$ is meromorphically extendable to $\overline{V(*)} \times B$ which is the closure in $\overline{E}$. Then we have

Theorem 2. Any Hamiltonian system which is holomorphic on E and meromorphically extendable to $\overline{E}$ must coincide to the Painlevé system (H_{VI}).

The theorem means that the Painlevé system (H_{VI}) is characterized by the pair of spaces $(E, \overline{E})$ with $E \subset \overline{E}$, and that a global analysis of the solutions of (H_{VI}) may be reduced to a geometry of $(E, \overline{E})$. We remark that the theorem is obtained without any assumptions on the singularities at $t = 0, 1, \infty$. We can remove the assumption of meromorphic extendability in the theorem, if, for any $t \in B$, any holomorphic function on a fiber $E(t)$ is meromorphically extendable to $\overline{E(t)}$.

§2. Review of the construction of the total space E

In this section, we review a part of the parer [7] by K. Okamoto, in a way suitable for our purpose, because, when the paper was written, it was not known that the Painlevé equations P_J are equivalent to the Painlevé systems (H_J).

2.1. The space $\overline{\Sigma}_\epsilon$. As a minimal compactification of $\mathbf{C}^2$ which is the phase space of the Painlevé system (H_{VI}), we take a 2-dimensional complex manifold $\overline{\Sigma}_\epsilon$ obtained by glueing four $U_i = \mathbf{C}^2 \ni (x_i, y_i), i = 0, 1, 2, 3$ via the following identification:

$$x_0 = x_1, \qquad y_0 = 1/y_1, \tag{2.1}$$

$$x_0 = 1/x_2, \qquad y_0 = x_2(\epsilon - x_2 y_2), \tag{2.2}$$

$$x_2 = x_3, \qquad y_2 = 1/y_3. \tag{2.3}$$

The manifold $\overline{\Sigma}_\epsilon$ is known as a Hirzebruch surface. It is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ if $\epsilon \neq 0$, and to a compactification of the cotangent bundle over $\mathbf{P}^1$ if $\epsilon = 0$ (see Example 2.16 in [6]). In the case of the sixth Painlevé system, the parameter ϵ is given by

$$\epsilon = \epsilon(+) = (\kappa_0 + \kappa_1 + \kappa_t - 1 + \kappa_\infty)/2. \tag{2.4}$$

(In [7], this compact manifold is denoted by Σ_ϵ. However, in the textbook [4], the same notation Σ_ϵ is used to denote an open manifold defined by U_0, U_2 and (2.2). Then we adopt the symbol $\overline{\Sigma}_\epsilon$ for our manifold in this series).

We consider each U_i or $U_i \times B$ as a coordinate neighborhood of $\overline{\Sigma}_\epsilon$ or of $\overline{\Sigma}_\epsilon \times B$ respectively. Notice that $y_1 = 0$ in U_1 corresponds to $y_3 = 0$ in U_3. In fact,

$$x_1 = 1/x_3, \qquad y_1 = y_3/[x_3(\epsilon y_3 - x_3)].$$

Considering the system (H_{VI}) as a Pfaffian system for three variables $x_0 = x,\ y_0 = y,\ t$ in $U_0 \times B$, we extend it to the whole space $\overline{\Sigma}_\epsilon \times B$ and we observe, in each $U_i \times B$, the foliation defined by the Pfaffian system. It is easy to see that, in $U_i \times B, i = 0, 2$, the foliation has no singularity and every leaf is transversal with fibers, because, not only in $U_0 \times B$ but also in $U_2 \times B$, the systems are written as

$$t(t-1)dx_i - P_i(x_i, y_i, t)dt = 0, \quad t(t-1)dy_i - Q_i(x_i, y_i, t)dt = 0,$$

where P_i and Q_i are certain polynomials of $x_i,\ y_i$ and t. However, in $U_i \times B, i = 1, 3$, there exist both singular points and vertical leaves. Recall that a vertical leaf is a one completely included in a fiber. For any $t \in B$, set

$$\begin{aligned}
D^{(0)}(t) &= (U_1(y_1 = 0) \times t) \cup (U_3(y_3 = 0) \times t) \cong \mathbf{P}^1, \\
a_\nu^{(0)}(t) &= \{(x_1, y_1, t) \mid x_1 = \nu, y_1 = 0\}, \qquad \nu = 0, 1, t, \\
a_\nu^{(0)}(t) &= \{(x_3, y_3, t) \mid x_3 = y_3 = 0\}, \qquad \nu = \infty,
\end{aligned}$$

where

$$U_i(y_i = 0) = \{(x_i, y_i) \in U_i \mid y_i = 0\}.$$

Then, $D^{(0)}(t) - \cup_\nu \{a_\nu^{(0)}(t)\}$ is a vertical leaf and the four points $a_\nu^{(0)}(t), \nu = 0, 1, t, \infty$ are singular points of the foliation. We remark that, in $\overline{\Sigma}_\epsilon$, every solution $(x(t), y(t))$ of (H_{VI}) can be holomorphically continued along any curve in B. This important fact follows from the Painlevé property for (H_{VI}) and from the fact that $\overline{\Sigma}_\epsilon$ is compact and the coordinate transformations among U_i are birational. Therefore, we see that every solution of (H_{VI}) does not pass through a point in $D^{(0)}(t) - \cup_\nu \{a_\nu^{(0)}(t)\}, t \in B$. However, as we see later, there are infinite number of solutions of (H_{VI}) which pass through the point $a_\nu^{(0)}(t), t \in B, \nu = 0, 1, t, \infty$.

2.2. The first quadratic transformations with centers $a_\nu^{(0)}(t), t \in B, \nu = 0, 1, t, \infty$. In the following two subsections, we see how to construct a fiber space $\overline{E}$ over B. Because the space $\overline{E}$ is defined as $\cup_{t \in B} \overline{E(t)} \times t$, we explain the construction of a fiber $\overline{E(t)}$ for any $t \in B$.

For any $t \in B$ and $\nu = 0, 1, t, \infty$, consider the quadratic transformation $Q_{a_\nu^{(0)}(t)}$ with center $a_\nu^{(0)}(t)$. Let $(z_\nu^{(1)}, w_\nu^{(1)}) \in \mathbf{C}^2$ and $(Z_\nu^{(1)}, W_\nu^{(1)}) \in \mathbf{C}^2$ be coordinate systems of $Q_{a_\nu^{(0)}(t)}(U_1 \times t)$ for $\nu = 0, 1, t$ or of $Q_{a_\infty^{(0)}(t)}(U_3 \times t)$ for $\nu = \infty$ defined by

$$
\begin{aligned}
x_1 &= \nu + z_\nu^{(1)}, \quad y_1 = z_\nu^{(1)} w_\nu^{(1)}, \\
x_1 &= \nu + Z_\nu^{(1)} W_\nu^{(1)}, \quad y_1 = W_\nu^{(1)},
\end{aligned}
\tag{2.5}
$$

for $\nu = 0, 1, t$, and

$$
\begin{aligned}
x_3 &= z_\infty^{(1)}, \quad y_3 = z_\infty^{(1)} w_\infty^{(1)}, \\
x_3 &= Z_\infty^{(1)} W_\infty^{(1)}, \quad y_3 = W_\infty^{(1)},
\end{aligned}
\tag{2.6}
$$

for $\nu = \infty$. In this series of papers, a letter with the superscript (i) denotes something which relates to the i-th quadratic transformation. We see that

$$
\begin{aligned}
D_\nu^{(1)}(t) :&= Q_{a_\nu^{(0)}(t)}(a_\nu^{(0)}(t)) \\
&= \{(z_\nu^{(1)}, w_\nu^{(1)}) \in \mathbf{C}^2 \mid z_\nu^{(1)} = 0\} \cup \{(Z_\nu^{(1)}, W_\nu^{(1)}) \in \mathbf{C}^2 \mid W_\nu^{(1)} = 0\}.
\end{aligned}
$$

By observing the Pfaffian system near $D_\nu^{(1)}(t)$, we can verify that the points

$$
\begin{aligned}
a_\nu^{(1)}(t) &= \{(Z_\nu^{(1)}, W_\nu^{(1)}) \mid Z_\nu^{(1)} = \kappa_\nu, W_\nu^{(1)} = 0\} \in D_\nu^{(1)}(t), \\
b_\nu^{(1)}(t) &= \{(z_\nu^{(1)}, w_\nu^{(1)}) \mid z_\nu^{(1)} = w_\nu^{(1)} = 0\} \in D_\nu^{(1)}(t),
\end{aligned}
$$

are singular points of the foliation, the latter point $b_\nu^{(1)}(t)$ is a point through which no solution of (H_{VI}) passes, and $D_\nu^{(1)}(t) - \{a_\nu^{(1)}(t), b_\nu^{(1)}(t)\}$ is a vertical leaf.

2.3. The second quadratic transformations with centers $a_\nu^{(1)}(t), t \in B, \nu = 0, 1, t, \infty$. Next, consider the quadratic transformation $Q_{a_\nu^{(1)}(t)}$ with center $a_\nu^{(1)}(t)$. Let $(z_\nu^{(2)}, w_\nu^{(2)}) \in \mathbf{C}^2$ and $(Z_\nu^{(2)}, W_\nu^{(2)}) \in \mathbf{C}^2$ be coordinate systems of $Q_{a_\nu^{(1)}(t)}(Q_{a_\nu^{(0)}(t)}(U_1 \times t))$ for $\nu = 0, 1, t$ or of $Q_{a_\infty^{(1)}(t)}(Q_{a_\infty^{(0)}(t)}(U_3 \times t))$ for $\nu = \infty$ defined by

$$
\begin{aligned}
Z_\nu^{(1)} &= \kappa_\nu + z_\nu^{(2)}, \qquad W_\nu^{(1)} = z_\nu^{(2)} w_\nu^{(2)}. \\
Z_\nu^{(1)} &= \kappa_\nu + Z_\nu^{(2)} W_\nu^{(2)}, \qquad W_\nu^{(1)} = W_\nu^{(2)}.
\end{aligned}
\tag{2.7}
$$

We see that

$$
\begin{aligned}
D_\nu^{(2)}(t) :&= Q_{a_\nu^{(1)}(t)}(a_\nu^{(1)}(t)) \\
&= \{(z_\nu^{(2)}, w_\nu^{(2)}) \in \mathbf{C}^2 \mid z_\nu^{(2)} = 0\} \cup \{(Z_\nu^{(2)}, W_\nu^{(2)}) \in \mathbf{C}^2 \mid W_\nu^{(2)} = 0\}.
\end{aligned}
$$

We can verify that the Pfaffian system is written as

$$\begin{aligned} t(t-1)dZ_\nu^{(2)} - P_\nu(Z_\nu^{(2)}, W_\nu^{(2)}, t)dt = 0, \\ t(t-1)dW_\nu^{(2)} - Q_\nu(Z_\nu^{(2)}, W_\nu^{(2)}, t)dt = 0, \end{aligned}$$

in the coordinates $Z_\nu^{(2)}$, $W_\nu^{(2)}$ where P_ν, Q_ν are certain polynomials of $Z_\nu^{(2)}, W_\nu^{(2)}$ and t. This means that the foliation has no singularity in $(Z_\nu^{(2)}, W_\nu^{(2)}, t)$-space $\mathbf{C}^2 \times B$ and every leaf in the space is transversal with fibers. On the other hand, in $(z_\nu^{(2)}, w_\nu^{(2)}, t)$-space, the point

$$b_\nu^{(2)}(t) = \{(z_\nu^{(2)}, w_\nu^{(2)}) \mid z_\nu^{(2)} = w_\nu^{(2)} = 0\}$$

is a singular point of the foliation through which no solution of (H_{VI}) passes and $D_\nu^{(1)}(t) - \{b_\nu^{(1)}(t), b_\nu^{(2)}(t)\}$ is a vertical leaf. Here, $D_\nu^{(1)}(t)$ and $b_\nu^{(1)}(t)$ denote also the proper images of themselves by $Q_{a_\nu^{(1)}(t)}$.

2.4. The space E. Let us denote by Φ_t the composition of all the first and the second quadratic transformations:

$$\Phi_t = \prod_{\nu=0,1,t,\infty} Q_{a_\nu^{(1)}(t)} \circ Q_{a_\nu^{(0)}(t)}.$$

Then we define $\overline{E(t)}$ and $\overline{E}$ by

$$\overline{E(t)} = \Phi_t(\Sigma_\epsilon \times t), \qquad \overline{E} = \bigcup_{t \in B} \overline{E(t)} \times t.$$

By following the above procedure, we see that the space $\overline{E}$ is a 3-dimensional complex manifold obtained by glueing

$$\begin{aligned} &\{(x_0, y_0, t) \in \mathbf{C}^2 \times B\}, \quad \{(x_2, y_2, t) \in \mathbf{C}^2 \times B\}, \\ &\{(x_1, y_1, t) \in \mathbf{C}^2 \times B \mid (x_1, y_1) \neq (0,0), (1,0), (t,0)\}, \\ &\{(x_3, y_3, t) \in \mathbf{C}^2 \times B \mid (x_3, y_3) \neq (0,0), (1,0), (1/t,0)\}, \\ &\{(z_\nu^{(1)}, w_\nu^{(1)}, t) \in \mathbf{C}^2 \times B \mid (z_\nu^{(1)}, w_\nu^{(1)}) \neq (0, 1/\kappa_\nu)\}, \\ &\{(Z_\nu^{(1)}, W_\nu^{(1)}, t) \in \mathbf{C}^2 \times B \mid (Z_\nu^{(1)}, W_\nu^{(1)}) \neq (\kappa_\nu, 0)\}, \\ &\{(z_\nu^{(2)}, w_\nu^{(2)}, t) \in \mathbf{C}^2 \times B\}, \quad \{(Z_\nu^{(2)}, W_\nu^{(2)}, t) \in \mathbf{C}^2 \times B\}, \ \nu = 0, 1, t, \infty, \end{aligned}$$

via the relations given in the above. Denote the proper images of $D^{(0)}(t)$,$D_\nu^{(1)}(t)$, $b_\nu^{(1)}(t)$, $\nu = 0, 1, t, \infty$ by the quadratic transformations by the same symbols, then the Pfaffian system on $\overline{E}$ defines a foliation on it with the following properties:

(i) $b_\nu^{(1)}(t)$ and $b_\nu^{(2)}(t)$ are singular points through which no solution of (H_{VI}) passes, for any $t \in B$ and ν,

(ii) $\overline{E} - \cup_{t\in B,\nu,i=1,2}\{b^{(i)}(t)\}$ is covered by complex one dimensional leaves which do not intersect with each other,

(iii) $D^{(0)}(t) - \cup_\nu\{b_\nu^{(1)}(t)\}$ and $D_\nu^{(1)}(t) - \cup_\nu\{b_\nu^{(1)}(t), b_\nu^{(2)}(t)\}$ are vertical leaves for any $t \in B$ and ν,

(iv) every leaf outside $\cup_{t\in B}(D^{(0)}(t)\cup(\cup_\nu D_\nu^{(1)}(t)))$ is an extension of a solution of (H_{VI}).

Lastly, define a space E by

$$E = \bigcup_{t\in B} E(t)\times t, \quad E(t) = \overline{E(t)} - D^{(0)}(t) \cup \bigcup_{\nu=0,1,t,\infty} D_\nu^{(1)}(t).$$

Then E has the properties (i),(ii),and (iii) stated in Introduction.

§3. Proof of Theorem 1.

We can easily verify that the space E is a 3-dimensional complex manifold obtained by glueing

$$\{(x_0, y_0, t) \in \mathbf{C}^2 \times B\}, \quad \{(x_2, y_2, t) \in \mathbf{C}^2 \times B\},$$
$$\{(Z_\nu^{(2)}, W_\nu^{(2)}, t) \in \mathbf{C}^2 \times B\}, \quad \nu = 0, 1, t, \infty,$$

via the coordinate transformations (2.1),(2.2),(2.3),(2.5),(2.6) and (2.7). We want to choose suitable coordinate systems for the six copies of $\mathbf{C}^2 \times B$ so that all coordinate transformations are symplectic.

We first notice that the transformation (2.2) is symplectic since

$$dy_0 \wedge dx_0 = dy_2 \wedge dx_2.$$

Therefore we set

$$(x(00), y(00)) = (x_0, y_0), \qquad (x(\infty 0+), y(\infty 0+)) = (x_2, y_2).$$

Next, we obtain, from (2.1),(2.5) and (2.7), the relation

$$x_0 = \nu + W_\nu^{(2)}(\kappa_\nu + Z_\nu^{(2)} W_\nu^{(2)}), \quad y_0 = 1/W_\nu^{(2)},$$

from which it follws that

$$dy_0 \wedge dx_0 = -dW_\nu^{(2)} \wedge dZ_\nu^{(2)},$$

for $\nu = 0, 1, t$. On the other hand, it follows from (2.3), (2.6) and (2.7) that

$$x_2 = W_\infty^{(2)}(\kappa_\infty + Z_\infty^{(2)} W_\infty^{(2)}), \quad y_2 = 1/W_\infty^{(2)},$$

which yields

$$dy_2 \wedge dx_2 = -dW_\infty^{(2)} \wedge dZ_\infty^{(2)}.$$

Therefore, by choosing new coordinate systems as

$$(x(1\infty), y(1\infty)) = (-Z_1^{(2)}, W_1^{(2)}), \qquad (x(t\infty), y(t\infty)) = (-Z_t^{(2)}, W_t^{(2)}),$$
$$(x(\infty 0-), y(\infty 0-)) = (-Z_\infty^{(2)}, W_\infty^{(2)}),$$

we obtain an expression of E given in Theorem 1.

§4. Proof of Theorem 2.

Let $\{K(*; x(*), y(*), t)\}_*$ be a Hamiltonian system holomorphic in E and meromorphically extendable to $\overline{E}$, namely, each $K(*; x(*), y(*), t)$ is holomorphic in $V(*) \times B$ and meromorphically extendable to its closure $\overline{V(*)} \times B$ in $\overline{E}$. For the sake of simplicity, we denote the variables $x(00), y(00)$ and the Hamiltonian $K(00; x(00), y(00), t)$ on $V(00) \times B$ by x, y and $K(x, y, t)$ respectively. In this section, we prove $K(x, y, t) = H_{VI}(x, y, t)$, which is the assertion of Theorem 2.

Let

$$K = \sum_{i,j \geq 0} a_{ij} x^i y^j,$$

be the Taylor expansion of K, where a_{ij} are holomorphic functions of t difined in B. By our assumption, the series (4.1) is convergent for every $x, y \in \mathbf{C}$ and $t \in B$.

4.1. Reduction of K to a polynomial. By recalling the construction of $\overline{E}$, we see that $\overline{V(00)} \times B$ contains a divisor $\{(x_1, y_1, t) \in \mathbf{C}^2 \times B \mid y_1 = 0, x_1 \neq 0, 1, t\}$ where $x = x_1, y = 1/y_1$. Therefore, by our assumption, $K(x_1, 1/y_1, t)$ must be meromorphic on $y_1 = 0, x_1 \neq 0, 1, t$, which implies that

$$a_{ij} = 0, \qquad j > M, \tag{4.1}$$

M being some nonnegative integer.

Denote by (X, Y, t) the coordinates $(x(\infty 0+), y(\infty 0+), t)$ of $V(\infty 0+)\times B$. Then, by (1.7), the Hamiltonian $K(\infty 0+)$ in $V(\infty 0+)\times B$ is given by

$$\begin{aligned} K(\infty 0+) &= \sum_{i,j\geq 0} a_{ij} X^{-(i-j)}(\epsilon - XY)^j \\ &\equiv \sum_{\mu\geq 1}\sum_{k=0}^{\mu-1}(-1)^k \frac{Y^k}{X^{\mu-k}} \sum_{j\geq k}\binom{j}{k}\epsilon^{j-k} a_{j+\mu,j}, \end{aligned}$$

where ϵ is given by (2.4) and $\equiv$ means mod power siries of X, Y with coefficients in the ring $\mathcal{O}(B)$ of functions holomorphic in B. Therefore we obtain

$$\sum_{j\geq 0} a_{j+\mu,j}\binom{j}{k}\epsilon^{j-k} = 0, \qquad k = 0, 1, \ldots, \mu - 1,$$

for every $\mu = 1, 2, \ldots$, because $K(\infty 0+)$ must be holomorphic on $X = 0$ by our assumption and $\{Y^k/X^{\mu-k}|\ \ \mu \geq 1, 0 \leq k \leq \mu - 1\}$ are linearly independent over $\mathcal{O}(B)$. We write a system of the linear equations as

$$(4.2)\qquad (a_{\mu,0}, a_{1+\mu,1}, \cdots)\left(\binom{p}{q}\epsilon^{p-q}\right)_{p\geq 0, 0\leq q\leq \mu-1} = (0, 0, \cdots, 0).$$

Noting

$$\det\left(\binom{p}{q}\epsilon^{p-q}\right)_{0\leq p,q\leq \mu-1} = 1,$$

and (4.1), we obtain, from (4.2) for every $\mu > M$, that $a_{ij} = 0$ for all i, j with $i - j > M$. Thus we have shown that $K(x, y)$ must be a polynomial of x, y with coefficients in $\mathcal{O}(B)$.

4.2. Conditions on the coefficients of K. In this subsection, we derive linear equations for the coefficients of K from the conditions that K is holomorphic in every coordinate neighborhood $V(*)\times B$ of E.

We first study the Hamiltonian $K(\infty 0-)$ in $V(\infty 0-)\times B$. Let $(X, Y, t) = (x(\infty 0-), y(\infty 0-), t)$, then, from (1.7) and (1.8), it follows

$$x = \frac{1}{Y(\kappa_\infty - XY)}, \quad y = Y(\kappa_\infty - XY)(\epsilon - (\kappa_\infty - XY)).$$

In the case where $\kappa_\infty \neq 0$, we have

$$
\begin{aligned}
K(\infty 0-) &= \sum_{i,j\geq 0} a_{ij} Y^{-(i-j)}(\kappa_\infty - XY)^{-(i-j)}(\epsilon - (\kappa_\infty - XY))^j \\
&= \sum_{\mu}\sum_{j\geq 0}\sum_{k\geq 0} \frac{(-)^k\binom{j}{k}\epsilon^{j-k}(\kappa_\infty - XY)^k}{Y^\mu(\kappa_\infty - XY)^\mu} a_{j+\mu,j} \\
&\equiv \sum_{\mu\geq 1}\sum_{k=0}^{\mu-1} \frac{(-1)^k}{Y^\mu(\kappa_\infty - XY)^{\mu-k}} \sum_{j\geq k}\binom{j}{k}\epsilon^{j-k} a_{j+\mu,j} \\
&\quad + \sum_{\mu\geq 1}\sum_{h=0}^{\mu-1} \frac{(-1)^\mu X^h}{Y^{\mu-h}} \sum_{j\geq 0}\phi_h^j(\mu) a_{j+\mu,j},
\end{aligned}
$$

where

$$
\phi_h^j(\mu) = \sum_{h+\mu\leq k\leq j} (-1)^{h+k-\mu}\binom{j}{k}\binom{k-\mu}{h}\epsilon^{j-k}{\kappa_\infty}^{k-h-\mu}.
$$

We see that

$$
\phi_h^j(\mu) = \begin{cases} 0, & j < h+\mu, \\ 1, & j = h+\mu, \\ (h+\mu+1)\epsilon - (h+1)\kappa_\infty, & j = h+\mu+1. \end{cases} \tag{4.3}
$$

Therefore we have the same system (4.2) and

$$
\sum_{j\geq 0} a_{j+\mu,j}\phi_h^j(\mu) = 0, \qquad h = 0,1,\ldots,\mu-1,
$$

and hence, we get a 2μ-system

(4.4)

$$
\begin{aligned}
&(a_{\mu,0}, a_{1+\mu,1},\cdots)\left(\left(\binom{p}{q}\epsilon^{p-q}\right)_{p\geq 0, 0\leq q\leq \mu-1}, \left(\phi_{q-\mu}^p(\mu)\right)_{p\geq 0,\mu\leq q\leq 2\mu-1}\right) \\
&\quad = (0,0,\cdots,0)
\end{aligned}
$$

In the case where $\kappa_\infty = 0$, we have

$$
K(\infty 0-) \equiv \sum_{\mu\geq 1}\sum_{k=0}^{2\mu-1} \frac{(-1)^\mu}{X^{\mu-k}Y^{2\mu-k}} \sum_{j\geq k} a_{j+\mu,j}\binom{j}{k}\epsilon^{j-k},
$$

and then we obtain a 2μ-system

$$
(a_{\mu,0}, a_{1+\mu,1},\cdots)\left(\binom{p}{q}\epsilon^{p-q}\right)_{p\geq 0, 0\leq q\leq 2\mu-1} = (0,0,\cdots,0), \tag{4.5}
$$

of which the first μ-system is equal to the system (4.2).

We next consider the Hamiltonian $K(0\infty)$. Let $(X, Y, t) = (x(0\infty), y(0\infty), t)$, then, from (1.4), namely, $x = Y(\kappa_0 - XY),\ y = 1/Y$, the Hamiltonian $K(0\infty)$ is given by

$$
\begin{aligned}
K(0\infty) &= \sum_{i,j\geq 0} a_{ij} Y^{-(j-i)}(\kappa_0 - XY)^i \\
&\equiv \sum_{\mu\geq 1}\sum_{k=0}^{\mu-1}(-1)^k \frac{X^k}{Y^{\mu-k}} \sum_{i\geq k}\binom{i}{k}\kappa_0^{i-k} a_{i,i+\mu}.
\end{aligned}
$$

Then, by the same argument in obtaing (4.2), we have a μ-system

$$
(4.6) \qquad (a_{0,\mu}, a_{1,1+\mu}, \cdots)\left(\binom{p}{q}\kappa_0{}^{p-q}\right)_{p\geq 0, 0\leq q\leq \mu-1} = (0, 0, \cdots, 0),
$$

for every $\mu = 1, 2, \ldots$. We note also

$$
\det\left(\binom{p}{q}\kappa_0{}^{p-q}\right)_{0\leq p,q\leq \mu-1} = 1.
$$

Lastly, consider the Hamiltonians $K(1\infty)$ and $K(t\infty)$. Let $(X, Y, t) = (x(1\infty), y(1\infty), t)$, then from (1.5): $x = 1 + Y(\kappa_1 - XY),\ y = 1/Y$, it follows that

$$
\begin{aligned}
K(1\infty) &= \sum_{i,j\geq 0} a_{ij}\frac{((1+\kappa_1 Y) - XY^2)^i}{Y^j} \\
&= \sum_{i,j\geq 0}\sum_{k\geq 0}(-1)^k\binom{i}{k}\frac{X^k}{Y^{j-2k}}(1+\kappa_1 Y)^{i-k} a_{ij}.
\end{aligned}
$$

Therefore if

$$
(4.7) \qquad a_{ij} = 0, \qquad j > M,
$$

$M \geq 2$ being an integer, then we have

$$
(4.8) \qquad \sum_{i\geq 0}\binom{i}{k} a_{iM} = 0, \qquad k = 0, 1, \ldots, [(M+1)/2] - 1,
$$

by observing the coefficients of $X^k/Y^{M-2k},\ k = 0, 1, \ldots, [(M+1)/2] - 1$. Here, [ ] denotes the Gauss symbol. Since the number $[(M+1)/2]$ often appears in this paper, we denote it by $\nu(M)$:

$$
\nu(M) = [(M+1)/2].
$$

By observing too the coefficients of X^k/Y^{M-1-2k}, $k=0,1,\dots,\nu(M-1)-1$, we have

$$\sum_{i\ge 0} a_{i,M-1}\binom{i}{k} + (k+1)\kappa_1 \sum_{i\ge 0} a_{iM}\binom{i}{k+1} = 0, \tag{4.9}$$

for $k = 0,1,\dots,\nu(M-1)-1$. Let $(X,Y,t) = (x(t\infty), y(t\infty), t)$,then the transformation between $(x,y,t)\in V(00)\times B$ and $(X,Y,t)\in V(t\infty)\times B$ is: $x = t + Y(\kappa_t - XY)$, $y = 1/Y$. Notice that it contains the time variable t explicitly, and the Hamiltonian $K(t\infty)$ in $V(t\infty)\times B$ is given by $K(t+Y(\kappa_t - XY), 1/Y, t) - 1/Y$. Under the same assumption (4.7), we obtain

$$\sum_{i\ge 0}\binom{i}{k} t^{i-k} a_{iM} = 0, \qquad k = 0,1,\dots,\nu(M)-1, \tag{4.10}$$

$$\sum_{i\ge 0} a_{i,M-1}\binom{i}{k} t^{i-k} + (k+1)\kappa_t \sum_{i\ge 0} a_{iM}\binom{i}{k+1} t^{i-(k+1)} = \delta_{M-1,1}, \tag{4.11}$$

$k=0,1,\dots,\nu(M-1)-1$, by observing the coefficients of X^k/Y^{M-2k}, $k = 0,1,\dots,\nu(M)-1$ and of X^k/Y^{M-1-2k}, $k=0,1,\dots,\nu(M-1)-1$, δ_{ij} denoting Kronecker's delta. By combining (4.8) and (4.10), we have a $2\nu(M)$-system

$$(a_{0M}, a_{1M}, \cdots) F(\infty, 2\nu(M)) = (0,0,\cdots,0), \tag{4.12}$$

where $F(\infty, 2n)$ is an $\infty \times 2n$ matrix $(f^i_j)_{i\ge 0, 0\le j\le 2n-1}$ with

$$f^p_q = \binom{p}{q}, \qquad p \ge 0,\ 0\le q\le n-1,$$
$$f^p_{q+n} = \binom{p}{q} t^{p-q}, \qquad p\ge 0,\ 0\le q\le n-1.$$

4.3. Reduction of K to a polynomial of small degree. The purpose of this subsection is to show

$$a_{ij} = 0, \qquad i > 3 \quad \text{or} \quad j > 2, \tag{4.13}$$

by proving

Proposition 4.1. For every $m\ge 2$, if $a_{ij} = 0$, for all i,j with i or $j > 3m$, then $a_{ij} = 0$, for all i,j with $i > 3m-3$ or $j > 2m-2$.

Assume that

$$a_{ij} = 0, \qquad i \quad \text{or} \quad j > 3m, \tag{4.14}$$

for an arbitrary but fixed $m \geq 2$.

We first notice that

$$a_{ij} = 0, \qquad i - j > m, \tag{4.15}$$

which is verified as follows. Let $m+1 \leq \mu \leq 3m$. Then, from the assumption (4.14), it follows that $a_{j+\mu,j} = 0$, $j > 2\mu - 1$. Consider the 2μ-system (4.4) or (4.5) for the 2μ unkowns $a_{j+\mu,j}$, $j = 0, 1, \dots, 2\mu - 1$. As is easily seen by (4.3), the determinant of the coefficient matrix of the system is 1, which yields $a_{j+\mu,j} = 0$, $j = 0, 1, \dots, 2\mu - 1$.

We now introduce a notion. By a *state* $S(k,l)$ of a polynomial Hamiltonian $K = \sum a_{ij} x^i y^j$, we mean a state

$$a_{ij} = 0, \qquad j > l \quad \text{or} \quad j - i > l - k.$$

Assume that K is in a state $S(k,l)$. Then $a_{i,i+(l-k)} = 0$ for $i > k$, $a_{ij} = 0$ for $j > l$, and $a_{il} = 0$ for $0 \leq i < k$ or $i > 3m$. Therefore, if $l - k \geq k + 1$, which means the number of equations is greater than or equal to that of unkowns, it follows from (4.6) that $a_{i,i+(l-k)} = 0$ for $0 \leq i \leq k$. In short, if $l \geq 2k+1$, then we can reduce $S(k,l)$ to $S(k+1,l)$ by using the linear system (4.6). We call the process Reduction A. On the other hand, if $2\nu(l) \geq 3m - k + 1$, we can reduce $S(k,l)$ to $S((k-1)^+, l-1)$ ($\alpha^+ = \max\{\alpha, 0\}$) by (4.12), the following Proposition 4.2, and the assumption $t \neq 0, 1$. We call the process Reduction B.

Proposition 4.2. Let $F_k(\infty, 2n)$ be a square matrix $(f^i_j)_{k \leq i \leq k+2n-1, 0 \leq j \leq 2n-1}$ which is a part of $F(\infty, 2n)$, then

$$\det F_k(\infty, 2n) = t^{nk}(t-1)^{n^2}. \tag{4.16}$$

Proof of Proposition 4.2. We can obtain

$$\det F_k(\infty, 2n) = t^n \det F_{k-1}(\infty, 2n),$$

for any $k \geq 1$, by virtue of a formula $\binom{p}{q} = \binom{p-1}{q} + \binom{p-1}{q-1}$. Therefore we have only to show (4.16) for $k = 0$. Let $I(t) = \det F_0(\infty, 2n)$, then $I(t)$ is a polynomial of t of degree at most n^2, the i-th derivative of $I(t)$ vanishes at $t = 1$ for every $i = 0, 1, \ldots, n^2 - 1$, and $I(0) = (-1)^{n^2}$. Then we have (4.16) for $k = 0$ and hence for general k.

We want to show that we can reduce a polynomial Hamiltonian K with (4.14) to the state $S(m, 2m)$ by a succesive use of Reductions A and B. We say that a state $S(k, l)$ is *reducible* if Reduction A or B is possible and it is *irreducible* if neither Reduction A nor B is possible. Then, a necessary and sufficient condition for a state $S(k, l)$ to be reducible is $l \geq 2k + 1$ or $2\nu(l) \geq 3m-k+1$, and hence $S(0, 3m)$ is reducible and $S(m, 2m)$ is irreducible.

Let us consider a set Σ of all states $S(k, l)$ such that

$$0 \leq k \leq 3m, \quad 2m \leq l \leq 3m, \quad l \geq 2k - 1, \quad l \geq 3m - k - 2.$$

We see that every state in Σ except $S(m, 2m)$ is reducible and Σ is *stable* under Reductions A and B, which means that every state in $\Sigma - \{S(m, 2m)\}$ is reduced to a state or states in Σ by Reductions A and B, by noting that Reduction A is impossible for $S(k, l)$ with $l = 2k-1$ or $l = 2k$ and Reduction B is impossible for it with $l = 3m - k - 2$ or $l = 3m - k - 1$.

We introduce a linear order $\succ$ in the set Σ by: $S(k, l) \succ S(k', l')$ if and only if $l > l'$, or $l = l'$ and $l - k > l' - k'$. Then we see that $S(0, 3m)$ is the highest state and $S(m, 2m)$ is the lowest one with respect to the order, and moreover, Reductions A and B reduce a state in $\Sigma - \{S(m, 2m)\}$ to strictly lower ones in Σ. By virtue of these properties, we can verify that there exists a chain of Reductions A and B which reduces $S(0, 3m)$ to $S(m, 2m)$. Thus we have proved that if K satisfies (4.14) then it does (4.15) and moreover it must be in the state $S(m, 2m)$.

In order to complete the proof of Proposition 4.1, we obtain a closed system of $4m+2$ linear equations for $4m+2$ unknowns $a_{i,2m-1}$, $m-1 \leq i \leq 3m-1$ and $a_{i,2m}$, $m \leq i \leq 3m$. We first have

$$a_{3m-1,2m-1} \;+\; m(\kappa_0 + \kappa_1 + \kappa_t - 1)a_{3m,2m} = 0,$$

from the last equation of system (4.4) for $\mu = m$ if $\kappa_\infty \neq 0$, or from that of (4.5) for $\mu = m$ if $\kappa_\infty = 0$. From the last equation of (4.6) for $\mu = m$, we obtain

$$a_{m-1,2m-1} \;+\; m\kappa_0 a_{m,2m} = 0.$$

On the other hand, we have, from (4.8) and (4.9) for $M = 2m$,

$$\sum_{i=m}^{3m} a_{i,2m}\binom{i}{k} = 0, \qquad k = 0, 1, \ldots, m-1,$$

$$\sum_{i=m-1}^{3m-1} a_{i,2m-1}\binom{i}{k} = 0, \qquad k = 0, 1, \ldots, m-2,$$

$$\sum_{i=m-1}^{3m-1} a_{i,2m-1}\binom{i}{m-1} + m\kappa_1 \sum_{i=m}^{3m} a_{i,2m}\binom{i}{m} = 0,$$

and, from (4.10) and (4.11) for $M = 2m$ with the condition $m \geq 2$,

$$\sum_{i=m}^{3m} a_{i,2m}\binom{i}{k} t^{i-k} = 0, \qquad k = 0, 1, \ldots, m-1,$$

$$\sum_{i=m-1}^{3m-1} a_{i,2m-1}\binom{i}{k} t^{i-k} = 0, \qquad k = 0, 1, \ldots, m-2,$$

$$\sum_{i=m-1}^{3m-1} a_{i,2m-1}\binom{i}{m-1} t^{i-(m-1)} + m\kappa_t \sum_{i=m}^{3m} a_{i,2m}\binom{i}{m} t^{i-m} = 0.$$

Thus we have obtained a $(4m+2)$-system

$$(4.17) \qquad \begin{aligned} &(a_{m-1,2m-1}, \ldots, a_{3m-1,2m-1}, a_{m,2m}, \ldots, a_{3m,2m}) G_m \\ &\qquad = (0, \ldots, 0), \end{aligned}$$

for $4m+2$ unknowns. Here a square matrix $G_n = (g^i_j(n))_{0 \leq i,j \leq 4m+1}$ for general n is defined by

$$g^0_0(n) = 1, \qquad g^{2m+1}_0(n) = m\kappa_0,$$

$$g^p_{q+1}(n) = \binom{p+(n-1)}{q}, \qquad 0 \leq p \leq 2m, \quad 0 \leq q \leq m-1,$$

$$g^{p+2m+1}_m(n) = m\kappa_1 \binom{p+n}{m}, \qquad 0 \leq p \leq 2m,$$

$$g^p_{q+m+1}(n) = \binom{p+(n-1)}{q} t^{p+(n-1)-q}, \quad 0 \leq p \leq 2m, \ 0 \leq q \leq m-1,$$

$$g^{p+2m+1}_{2m}(n) = m\kappa_t \binom{p+n}{m} t^{p+n-m}, \qquad 0 \leq p \leq 2m,$$

$$g^{p+2m+1}_{q+2m+1}(n) = \binom{p+n}{q}, \qquad 0 \leq p \leq 2m, \quad 0 \leq q \leq m-1,$$

$$g^{p+2m+1}_{q+3m+1}(n) = \binom{p+n}{q} t^{p+n-q}, \qquad 0 \leq p \leq 2m, \quad 0 \leq q \leq m-1,$$

$$g^{2m}_{4m+1}(n) = 1, \qquad g^{4m+1}_{4m+1}(n) = m(\kappa_0 + \kappa_1 + \kappa_t - 1),$$

$$g^i_j(n) = 0, \qquad \text{for other } i, j.$$

Then we obtain that $a_{i,2m-1} = 0,\ m-1 \leq i \leq 3m-1$, and $a_{i,2m} = 0,\ m \leq i \leq 3m$, by (4.17) and the following Proposition 4.3.

Proposition 4.3.

$$\det G_n = -mt^{2mn}(t-1)^{2m^2}. \tag{4.18}$$

Proof of Proposition 4.3. We see that

$$\det G_n = t^{2m} \det G_{n-1},$$

for any $n \geq 1$. Therefore we have only to show (4.18) for $n = 0$. Let $J(t) = \det G_0$. We can verify that all the second order partial derivatives of $J(t)$ with respect to the parameters $\kappa_\nu,\ \nu = 0, 1, t$ are identically zero and all the fisrt order partial derivabives with respect to these parameters vanish at $\kappa_0 = \kappa_1 = \kappa_t = 0$. Then $J(t)$ is independent of these parameters. It is easy to see that $J(t)|_{\kappa_0=\kappa_1=\kappa_t=0} = -m(t-1)^{2m^2}$ by virtue of Proposition 4.2, which completes the proof of Proposition 4.3.

Thus we have shown Proposition 4.1 and hence (4.13).

4.4. Completion of the proof of Theorem 2. We can now assume (4.13). From (4.4) or (4.5) for $\mu = 3, 2, 1$, we have

$$a_{30} = a_{20} = a_{31} = 0,$$
$$a_{10} + \epsilon a_{21} + \epsilon^2 a_{32} = 0,$$
$$a_{21} + (2\epsilon - \kappa_\infty) a_{32} = 0.$$

From (4.6) for $\mu = 2, 1$, it follows that

$$a_{02} = 0,$$
$$a_{01} + \kappa_0 a_{12} = 0.$$

On the other hand, by (4.8),(4.9),(4.10) and (4.11) for $M = 2$, we have

$$a_{12} + a_{22} + a_{32} = 0,$$
$$a_{01} + a_{11} + a_{21} + \kappa_1 a_{12} + 2\kappa_1 a_{22} + 3\kappa_1 a_{32} = 0,$$
$$t a_{12} + t^2 a_{22} + t^3 a_{32} = 0,$$
$$a_{01} + t a_{11} + t^2 a_{21} + \kappa_t a_{12} + 2\kappa_t t a_{22} + 3\kappa_t t^2 a_{32} = 1.$$

We remark that the system of the above equations is a necessary and sufficient condition for our Hamiltonian K with (4.13) to define a holomorphic Hamiltonian system on E.

It is easy to verify that the 7-system for $a_{32}, a_{22}, a_{12}, a_{21}, a_{11}, a_{01}$ and a_{10} has a unique solution

$$\begin{aligned}
&a_{32} = \frac{1}{t(t-1)}, \qquad a_{22} = \frac{-(t+1)}{t(t-1)}, \qquad a_{12} = \frac{t}{t(t-1)},\\
&a_{21} = \frac{-(\kappa_0+\kappa_1+\kappa_t-1)}{t(t-1)}, \qquad a_{11} = \frac{(\kappa_0+\kappa_1)t+(\kappa_0+\kappa_t-1)}{t(t-1)},\\
&a_{01} = \frac{-\kappa_0 t}{t(t-1)}, \qquad a_{10} = \frac{\kappa}{t(t-1)},
\end{aligned}$$

by $t \neq 0, 1$. Thus we have completed the proof of Theorem 2.

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