Bäcklund Transformations and the Manifolds of Painlevé Systems

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1   Introduction

The purpose of this paper is to show that we can take coordinate systems determined by the Bäcklund transformations as coordinate systems of the manifolds of Painlevé systems constructed by K. Okamoto ([10]) (except the first one) and that the manifolds with parameters equivalent under the corresponding affine Weyl groups are mutually isomorphic.

The J-th Painlevé system ($J = II, III, IV, V, VI$) which is equivalent to the J-th Painlevé equation is the following Hamiltonian system

$$(H_{J,\alpha}) \qquad \delta q = \{H_J(q, p, t, \alpha), q\}, \quad \delta p = \{H_J(q, p, t, \alpha), p\},$$

where $\delta = d/dt$ for $J = II, IV$, $\delta = td/dt$ for $J = III, V$, $\delta = t(t-1)d/dt$ for $J = VI$, $\{\cdot, \cdot\}$ is the Poisson bracket defined by

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p},$$

and the Hamiltonian $H_J(q, p, t, \alpha)$, $\alpha = (\alpha_0, \alpha_1, ...)$ being parameters with a relation, is given by

$$
\begin{aligned}
H_{II}(q, p, t, \alpha) &= \frac{1}{2}p^2 - (q^2 + \frac{t}{2})p - \alpha_1 q \\
&\quad (\alpha_0 + \alpha_1 = 1), \\
H_{III}(q, p, t, \alpha) &= q^2 p(p-1) + q[(\alpha_0 + \alpha_2)p - \alpha_0] + tp \\
&\quad (\alpha_0 + 2\alpha_1 + \alpha_2 = 1), \\
H_{IV}(q, p, t, \alpha) &= qp(p - q - 2t) - 2\alpha_1 p - 2\alpha_2 q \\
&\quad (\alpha_0 + \alpha_1 + \alpha_2 = 1), \\
H_V(q, p, t, \alpha) &= q(q-1)p(p+t) - (\alpha_1 + \alpha_3)qp + \alpha_1 p + \alpha_2 tq \\
&\quad (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1),
\end{aligned}
$$

$$
\begin{aligned}
H_{VI}(q,p,t,\alpha) \;=\;& q(q-1)(q-t)p^2 - [(\alpha_0-1)q(q-1) + \alpha_4(q-1)(q-t) \\
& + \alpha_3 q(q-t)]p + \alpha_2(\alpha_1+\alpha_2)(q-t) \\
& (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1).
\end{aligned}
$$

We notice that the forms of the Hamiltonians for $J = III, IV, V$ given here are slightly different from those in [2],[14],[4]. The Hamiltonian for $J = III$ or IV or V is obtained from that for $J = III'$ or IV or V in [2] respectively by certain change of variables (see Section 3).

Each Painlevé system determines a complex one dimensional nonsingular foliation of $\mathbf{C}^2 \times B_J (\ni (q,p,t))$ where

$$
B_{II} = B_{IV} = \mathbf{C}, \quad B_{III} = B_V = \mathbf{C} - \{0\}, \quad B_{VI} = \mathbf{C} - \{0,1\}.
$$

The system is holomorphically extended to one on a manifold $E_{J,\alpha}$ which is a fiber space over B_J having the $\mathbf{C}^2 \times B_J$ as a fiber subspace and the extended system defines a *uniform* foliation $\mathcal{F}_{J,\alpha}$ of $E_{J,\alpha}$ although the foliation of the $\mathbf{C}^2 \times B_J$ is not uniform ([14],[4],[10]). Here the *uniformity* of the foliation $\mathcal{F}_{J,\alpha}$ means that, *for any point $P_0 \in E_{J,\alpha}$, every curve in B_J starting from $\pi_J(P_0)$ is lifted on the leaf passing through P_0, where π_J is the projection from $E_{J,\alpha}$ to B_J.* We notice that the *uniformity* of the foliation is equivalent to the so-called *Painlevé property* for the Painlevé system, that is, *if $(q(t), p(t))$ is a local solution of $(H_{J,\alpha})$ determined by an arbitrary initial condition $q(t_0) = q_0 \in \mathbf{C}$, $p(t_0) = p_0 \in \mathbf{C}$ with $t_0 \in B_J$, then both $q(t)$ and $p(t)$ can be meromorphically continued along any curve in B_J with a starting point t_0.* The fibers of $E_{J,\alpha}$ are called the *spaces of initial conditions*([10]). Each $E_{J,\alpha}$ is described by the original chart $\mathbf{C}^2 \times B_J$ and a finite number of copies $\mathbf{C}_i^2 \times B_J$ of $\mathbf{C}^2 \times B_J$ where coordinate transformations are certain birational symplectic ones ([14],[4]).

On the other hand, each Painlevé system admits a Bäcklund transformation group of certain birational symplectic transformations each of which preserves the form of the Hamiltonian and changes the parameters α_i as an element of an affine Weyl group ([6],[7],[8],[9]). This fact was first recognized by K. Okamoto ([11]), but our presentation in the following is different from his.

Let $K = \mathbf{C}(q,p,t,\alpha)$ ($\alpha = (\alpha_0, \alpha_1, ...)$) be a differential field of rational functions of q, p, t, α with a derivation δ defined by

$$
\delta f = \frac{\partial f}{\partial q} \cdot \{H_J(q,p,t,\alpha), q\} + \frac{\partial f}{\partial p} \cdot \{H_J(q,p,t,\alpha), p\} + \delta' f, \quad f \in K,
$$

where δ' is $\partial/\partial t$ for $J = II, IV$, $t\partial/\partial t$ for $J = III, V$, and $t(t-1)\partial/\partial t$ for $J = VI$. (Notice that $\delta\alpha_i = 0$.) Then, there is a Bäcklund transformation group W which is a lift of an affine Weyl group acting on the α-space such that

(i) each $w \in W$ is an isomorphism from the field K to itself,
(ii) $\delta w = w\delta$, for $w \in W$,
(iii) $w\{f,g\} = \{w(f), w(g)\}$ for $w \in W$, $f, g \in K$.

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The group W is generated by a finite number of reflections s_i.

For $w \in W$, consider a birational symplectic change of variables from (q, p, t) to (q_w, p_w, t_w) defined by

$$q_w = w(q), \quad p_w = w(p), \quad t_w = w(t).$$

Then the Hamiltonian system $(H_{J,\alpha})$ with $\alpha_0 + ... = 1$ is transformed to $(H_{J,w(\alpha)})$:

$$\delta q_w = \{H_J(q_w, p_w, t_w, w(\alpha)), q_w\}, \quad \delta p_w = \{H_J(q_w, p_w, t_w, w(\alpha)), p_w\},$$

where $w(\alpha) = (w(\alpha_0), w(\alpha_1), ...)$ with $w(\alpha_0) + ... = 1$. (We notice that $t_w = t$ for every $w \in W$ in the case of $J \neq III$ and $t_w = \pm t$ in the case of $J = III$ and $\delta = \delta_w$ where δ_w is the derivation with respect to t_w.) Hence w extends the domain of definition $\mathbf{C}^2 \times B_J$ of the system $(H_{J,\alpha})$ to $\mathbf{C}^2 \times B_J \sqcup \mathbf{C}_w^2 \times B_{J,w}/ \sim$, where $\sim$ is an identification of the points $(q, p, t) \in \mathbf{C}^2 \times B_J$ and $(q_w, p_w, t_w) \in \mathbf{C}_w^2 \times B_{J,w}(\simeq \mathbf{C}^2 \times B_J)$ by the above relation. The system $(H_{J,w(\alpha)})$ is considered to be the restriction of the extended Hamiltonian system on the chart $\mathbf{C}_w^2 \times B_{J,w}$.

We extend the domain of definition $\mathbf{C}^2 \times B_J$ of $(H_{J,\alpha})$ by all $w \in W$. Let $E_{J,\alpha}^W$ be a manifold obtained by gluing the copies $\mathbf{C}_w^2 \times B_{J,w}, w \in W$ of $\mathbf{C}^2 \times B_J$ via the relations

$$q_{w'} = w'w^{-1}(q_w), \quad p_{w'} = w'w^{-1}(p_w), \quad t_{w'} = w'w^{-1}(t_w)$$

for any $w, w' \in W$:

$$E_{J,\alpha}^W := \left(\bigsqcup_{w \in W} \mathbf{C}_w^2 \times B_{J,w} \right) / \sim .$$

The identification $\sim$ is well defined since W is a group. We often consider each $\mathbf{C}_w^2 \times B_{J,w}$ a subset of $E_{J,\alpha}^W$.

The manifold $E_{J,\alpha}^W$ is a fiber space over B_J and the extension of the Painlevé system $(H_{J,\alpha})$ on $E_{J,\alpha}^W$ defines a complex one dimensional nonsingular foliation of $E_{J,\alpha}^W$ each leaf of which is transversal to fibers.

The main result of this paper is stated as:

Theorem 1. *The identity mapping ϕ from $\mathbf{C}^2 \times B_J \subset E_{J,\alpha}^W$ to the original chart $\mathbf{C}^2 \times B_J$ of $E_{J,\alpha}$ can be extended to an isomorphism*

$$\varphi : \quad E_{J,\alpha}^W \longrightarrow E_{J,\alpha}.$$

In general, for any $w \in W$, the mapping ϕ_w from the chart $\mathbf{C}_w^2 \times B_{J,w} \ni (q_w, p_w, t_w)$ of $E_{J,\alpha}^W$ to the original chart $\mathbf{C}^2 \times B_J \ni (q, p, t)$ of $E_{J,w(\alpha)}$ defined by $(q, p, t) = (q_w, p_w, t_w)$ can be extended to an isomorphism

$$\varphi_w : \quad E_{J,\alpha}^W \longrightarrow E_{J,w(\alpha)}.$$

Here an isomorphism means a biholomorphic mapping which preserves fibers and leaves of the foliations.

In the proof of the theorem, the uniformity of the foliation $\mathcal{F}_{J,\alpha}$ of $E_{J,\alpha}$ plays an essential role. One can find a proof of the uniformity in [15],[18],[1], for example. By means of the theorem, we can say that the manifold $E_{J,\alpha}$ is covered by the coordinate systems $\mathbf{C}_w^2 \times B_{J,w}, w \in W$. The coordinate systems are convenient in that the Hamiltonians on them are easily obtained by the changes of parameters. The following important fact is also an immediate consequence of the theorem.

Corollary. *The manifolds $E_{J,\alpha}$ and $E_{J,\alpha'}$ are isomorphic if there exists $w \in W$ such that $\alpha' = w(\alpha)$.*

In a private communication, we were informed that H. Umemura and J. Matsuzawa had also obtained the corollary.

We notice that the manifold $E_{J,\alpha}^W$ is covered by a finite number of coordinate systems although it is defined by infinitely many ones. The fact is verified by the above corollary, the following theorem in which s_i are the generators of W, and the property that, for any α, there is a $w \in W$ such that none of $w(\alpha_i)$ (and $w(\alpha_1 + \alpha_2)$ for $J = VI$) vanish. The theorem is also used in the proof of Theorem 1.

Theorem 2. (The case of $J = II, III, IV, V$) *If none of α_i vanish, then*

$$\left(\mathbf{C}^2 \times B_J \bigsqcup \left(\bigsqcup_i \mathbf{C}_{s_i}^2 \times B_{J,s_i} \right) \right) / \sim \; \simeq E_{J,\alpha}.$$

(The case of $J = VI$) *If none of α_i and $\alpha_1 + \alpha_2$ vanish, then*

$$\left(\mathbf{C}^2 \times B_{VI} \bigsqcup \left(\bigsqcup_{i=0,2,3,4} \mathbf{C}_{s_i}^2 \times B_{VI} \right) \bigsqcup \mathbf{C}_{s_1 s_2}^2 \times B_{VI} \right) / \sim \; \simeq E_{VI,\alpha}.$$

In Section 2, we give lists of certain generators of Bäcklund transformation groups of Painlevé systems and show some propositions which will be used in the proof of Theorem 1. In Section 3, we review the descriptions of the manifolds $E_{J,\alpha}$ ([14],[4]) and give lists of Hamiltonians on all charts and then we show a proposition. The succeeding sections are devoted to proving Theorems 1 and 2. We first prove Theorem 2 in Section 4 and then prove Theorem 1 in Sections 5 and 6. In the case of $J = VI$, there appear divisors in $E_{J,\alpha}^W$ and a divisor in $E_{J,\alpha}$ at infinity of the original chart which are invariant with respect to the foliations, and hence we have to observe them precisely.

In the end of this section, we note a work by H. Watanabe in which he has given some relations between Bäcklund transformations and suitable descriptions of the manifolds ([16],[17]).

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2   Bäcklund transformation groups

In this section, we give explicit forms of some natural generators s_i of the Bäcklund transformation group W of each Painlevé system and some propositions. We give also generators of the extended Bäcklund transformation group $\widetilde{W}$ although it is not used in this paper. Each list consists of the type of affine Weyl group, Dynkin diagram, generalized Cartan matrix, the fundamental relations of the generators of the Bäcklund transformation group W and the extended Bäcklund transformation group $\widetilde{W}$, and the explicit forms of the generators. Except for the case of P_{III} the group $\widetilde{W}$ is the full symmetry group which preserves the independent variable t.

2.1   The case of $J = II$

$$A_1^{(1)}: \quad \begin{matrix} \alpha_0 & \alpha_1 \\ \circ \Leftrightarrow \circ \end{matrix} \quad (\alpha_0 + \alpha_1 = 1) \quad A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$W(A_1^{(1)}) = \langle s_0, s_1 \rangle: \quad s_0^2 = s_1^2 = 1.$$

$$\widetilde{W}(A_1^{(1)}) = \langle s_0, s_1, \pi \rangle: \quad s_0^2 = s_1^2 = 1 \; ; \; \pi^2 = 1, \; \pi s_0 = s_1 \pi, \; \pi s_1 = s_0 \pi.$$

	α_0	α_1	q	p
s_0	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$q + \frac{\alpha_0}{p-2q^2-t}$	$p + \frac{4\alpha_0 q}{p-2q^2-t} + \frac{2\alpha_0^2}{(p-2q^2-t)^2}$
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$q + \frac{\alpha_1}{p}$	p
π	α_1	α_0	$-q$	$-p + 2q^2 + t$

The last list must be read as

$$s_0(\alpha_0) = -\alpha_0, \quad s_0(\alpha_1) = \alpha_1 + 2\alpha_0,$$

$$s_0(q) = q + \frac{\alpha_0}{p - 2q^2 - t}, \quad s_0(p) = p + \frac{4\alpha_0 q}{p - 2q^2 - t} + \frac{2\alpha_0^2}{(p - 2q^2 - t)^2},$$

and so on.

2.2   The case of $J = III$

$$C_2^{(1)}: \quad \begin{matrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \circ \Rightarrow \circ \Leftarrow \circ \end{matrix} \quad (\alpha_0 + 2\alpha_1 + \alpha_2 = 1) \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$W(C_2^{(1)}) = \langle s_0, s_1, s_2 \rangle: \quad s_0^2 = s_1^2 = s_2^2 = 1, \quad (s_0 s_1)^4 = (s_1 s_2)^4 = 1.$$

$$\widetilde{W}(C_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle: \quad \begin{aligned} & s_0^2 = s_1^2 = s_2^2 = 1, \quad (s_0 s_1)^4 = (s_1 s_2)^4 = 1, \\ & \pi^2 = 1, \; \pi s_0 = s_2 \pi, \; \pi s_1 = s_1 \pi, \; \pi s_2 = s_0 \pi. \end{aligned}$$

	α_0	α_1	α_2	t	q	p
s_0	$-\alpha_0$	$\alpha_1+\alpha_0$	α_2	t	$q+\frac{\alpha_0}{p}$	p
s_1	$\alpha_0+2\alpha_1$	$-\alpha_1$	$\alpha_2+2\alpha_1$	$-t$	q	$p-\frac{2\alpha_1}{q}+\frac{t}{q^2}$
s_2	α_0	$\alpha_1+\alpha_2$	$-\alpha_2$	t	$q+\frac{\alpha_2}{p-1}$	p
π	α_2	α_1	α_0	$-t$	$-q$	$1-p$

We remark that the Bäcklund transformations of the Hamiltonian system (H_{III}) can also be described in terms of an extension of the affine Weyl group $W(A_1^{(1)}) \times W(A_1^{(1)})$. In this paper, however, we make use of $W(C_2^{(1)})$ for convenience, since it is directly related to the description of the manifold $E_{III,\alpha}$ given in the next section.

2.3 The case of $J = IV$

$$A_2^{(1)}: \qquad \overset{\alpha_0}{\underset{\alpha_1 \quad\quad \alpha_2}{\triangle}} \qquad (\alpha_0+\alpha_1+\alpha_2=1) \qquad A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$W(A_2^{(1)}) = \langle s_0, s_1, s_2 \rangle: \quad s_0^2 = s_1^2 = s_2^2 = 1, \quad (s_0 s_1)^3 = (s_1 s_2)^3 = (s_2 s_0)^3 = 1.$$

$$\widetilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle: \quad \begin{matrix} s_0^2 = s_1^2 = s_2^2 = 1, \quad (s_0 s_1)^3 = (s_1 s_2)^3 = (s_2 s_0)^3 = 1, \\ \pi^3 = 1, \ \pi s_0 = s_1 \pi, \ \pi s_1 = s_2 \pi, \ \pi s_2 = s_0 \pi. \end{matrix}$$

	α_0	α_1	α_2	q	p
s_0	$-\alpha_0$	$\alpha_1+\alpha_0$	$\alpha_2+\alpha_0$	$q+\frac{2\alpha_0}{p-q-2t}$	$p+\frac{2\alpha_0}{p-q-2t}$
s_1	$\alpha_0+\alpha_1$	$-\alpha_1$	$\alpha_2+\alpha_1$	q	$p-\frac{2\alpha_1}{q}$
s_2	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$q+\frac{2\alpha_2}{p}$	p
π	α_1	α_2	α_0	$-p$	$-p+q+2t$

2.4 The case of $J = V$

$$A_3^{(1)}: \qquad \overset{\alpha_0}{\underset{\alpha_2}{\underset{\alpha_1 \quad\quad\quad \alpha_3}{\diamond}}} \qquad (\alpha_0+\alpha_1+\alpha_2+\alpha_3=1) \qquad A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$W(A_3^{(1)}) = \langle s_0, s_1, s_2, s_3 \rangle: \quad s_i^2 = 1, \quad (s_i s_{i+2})^2 = 1, \quad (s_i s_{i+1})^3 = 1.$$

$$\widetilde{W}(A_3^{(1)}) = \langle s_0, s_1, s_2, s_3, \pi \rangle: \quad \begin{matrix} s_i^2 = 1, \quad (s_i s_{i+2})^2 = 1, \quad (s_i s_{i+1})^3 = 1, \\ \pi^4 = 1, \ \pi s_i = s_{i+1} \pi. \quad (i \in \mathbf{Z}/4\mathbf{Z}) \end{matrix}$$

	α_0	α_1	α_2	α_3	q	p
s_0	$-\alpha_0$	$\alpha_1+\alpha_0$	α_2	$\alpha_3+\alpha_0$	$q+\frac{\alpha_0}{p+t}$	p
s_1	$\alpha_0+\alpha_1$	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	q	$p-\frac{\alpha_1}{q}$
s_2	α_0	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$q+\frac{\alpha_2}{p}$	p
s_3	$\alpha_0+\alpha_3$	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	q	$p-\frac{\alpha_3}{q-1}$
π	α_1	α_2	α_3	α_0	$-\frac{p}{t}$	$(q-1)t$

2.5 The case of $J = VI$

$D_4^{(1)}:$
$\begin{array}{c}\alpha_0 \\ \alpha_1\end{array} \!\!\!>\!\!\!\!\!-\!\!\!\!\!<\!\!\!\begin{array}{c}\alpha_3 \\ \alpha_4\end{array}$ with α_2 in the center $\quad (\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1)$

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$W(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4 \rangle : \quad s_i^2 = s_2^2 = 1, \quad (s_i s_j)^2 = 1, \quad (s_i s_2)^3 = 1.$$

$$\begin{aligned}
\widetilde{W}(D_4^{(1)}) &= \langle s_0, s_1, s_2, s_3, s_4, \sigma_{01|34}, \sigma_{03|14}, \sigma_{04|13} \rangle : \\
&\quad s_i^2 = s_2^2 = 1, \quad (s_i s_j)^2 = 1, \quad (s_i s_2)^3 = 1, \quad (i, j \neq 2) \\
&\quad \sigma_{01|34}(s_0, s_1, s_2, s_3, s_4) = (s_1, s_0, s_2, s_4, s_3)\sigma_{01|34}, \\
&\quad \sigma_{03|14}(s_0, s_1, s_2, s_3, s_4) = (s_3, s_4, s_2, s_0, s_1)\sigma_{03|14}, \\
&\quad \sigma_{04|13}(s_0, s_1, s_2, s_3, s_4) = (s_4, s_3, s_2, s_1, s_0)\sigma_{04|13}.
\end{aligned}$$

The Diagram automorphisms $\sigma_{01|34}, \sigma_{03|14}, \sigma_{04|13}$ generate the Klein group of order 4.

	α_0	α_1	α_2	α_3	α_4	q	p
s_0	$-\alpha_0$	α_1	$\alpha_2+\alpha_0$	α_3	α_4	q	$p-\frac{\alpha_0}{q-t}$
s_1	α_0	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	α_4	q	p
s_2	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$\alpha_4+\alpha_2$	$q+\frac{\alpha_2}{p}$	p
s_3	α_0	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	α_4	q	$p-\frac{\alpha_3}{q-1}$
s_4	α_0	α_1	$\alpha_2+\alpha_4$	α_3	$-\alpha_4$	q	$p-\frac{\alpha_4}{q}$
$\sigma_{01|34}$	α_1	α_0	α_2	α_4	α_3	$\frac{t(t-1)}{q-t}+t$	$-\frac{(q-t)((q-t)p+\alpha_2)}{t(t-1)}$
$\sigma_{03|14}$	α_3	α_4	α_2	α_0	α_1	$\frac{t}{q}$	$-\frac{q(qp+\alpha_2)}{t}$
$\sigma_{04|13}$	α_4	α_3	α_2	α_1	α_0	$-\frac{t-1}{q-1}+1$	$\frac{(q-1)((q-1)p+\alpha_2)}{t-1}$

2.6 Propositions

Recall the definition of the manifold $E_{J,\alpha}^W$ by gluing the copies $\mathbf{C}_w^2 \times B_{J,w}, w \in W$ of $\mathbf{C}^2 \times B_J$ via the identification determined by the Bäcklund transformations. We first give a proposition concerning the extension of the domain of definition, which will be used in the proof of Theorem 1.

We see that the Hamiltonian system

$$\delta q_w = \{H_J(q_w, p_w, t_w, w(\alpha)), p_w\}, \quad \delta p_w = \{H_J(q_w, p_w, t_w, w(\alpha)), q_w\}$$

on $\mathbf{C}_w^2 \times B_{J,w}$ is changed to the Hamiltonian system

$$\delta q_{ws} = \{H_J(q_{ws}, p_{ws}, t_{ws}, ws(\alpha)), p_{ws}\}, \quad \delta p_{ws} = \{H_J(q_{ws}, p_{ws}, t_{ws}, ws(\alpha)), q_{ws}\}$$

on $\mathbf{C}_{ws}^2 \times B_{J,ws}$, where $w \in W$ and s is a generator of W. Let us denote by $D_{w,ws} \subset \mathbf{C}_{ws}^2 \times B_{J,ws}$ the divisor defined as the complement of $\mathbf{C}_w^2 \times B_{J,w}$: $D_{w,ws} = \mathbf{C}_{ws}^2 \times B_{J,ws} - \mathbf{C}_w^2 \times B_{J,w}$. We notice that $D_{w,ws}$ can be an empty set. Then we have

Proposition 2.1. *In the case of $J \neq VI$, every divisor $D_{w,ws}$ is transversal to leaves. In the case of $J = VI$, every divisor $D_{w,ws}$ ($s \neq s_2$) is transversal to leaves, however the divisor D_{w,ws_2} ($w(\alpha_2) \neq 0$) is invariant with respect to the foliation if $w(\alpha_1) = 0$.*

Proof. The proposition is verified by observing the Hamiltonian system on $\mathbf{C}_{ws}^2 \times B_{J,ws}$. For example, consider first the case of $J = II, s = s_1$. Since $q_w = q_{ws} - w(\alpha_1)/p_{ws}, p_w = p_{ws}$, we have $D_{w,ws} = \{p_{ws} = 0\}$ if $w(\alpha_1) \neq 0$. (Notice that if $w(\alpha_1) = 0$, then $D_{w,ws} = \emptyset$.) By $\delta p_{ws} = \{H_{II}(q_{ws}, p_{ws}, t, ws(\alpha)), p_{ws}\}$ and $\{H_{II}(q_{ws}, p_{ws}, t, ws(\alpha)), p_{ws}\}|_{p_{ws}=0} = -w(\alpha_1)$, we see that $D_{w,ws}$ is transversal to leaves. We consider next the case of $J = VI, s = s_2$. In the case, we have $D_{w,ws} = \{p_{ws} = 0\}$ if $w(\alpha_2) \neq 0$, $\delta p_{ws} = \{H_{VI}(q_{ws}, p_{ws}, t, ws(\alpha)), p_{ws}\}$, and $\{H_{VI}(q_{ws}, p_{ws}, t, ws(\alpha)), p_{ws}\}|_{p_{ws}=0} = w(\alpha_2)w(\alpha_1)$, and then we obtain the last assertion.

The following proposition will also be used in the proof of Theorem 1.

Proposition 2.2. *In the case of $J \neq VI$, for any α, there is a $w \in W$ such that $w(\alpha_i) \neq 0$ for all i. In the case of $J = VI$, for any α, there is a $w \in W$ such that $w(\alpha_1 + \alpha_2) \neq 0$ and $w(\alpha_i) \neq 0$ for all i.*

Proof. This fact follows from the actions of translation operators contained in the affine Weyl group W with respect to the root lattice.

3   The manifolds $E_{J,\alpha}$

In this section, we give descriptions of the manifolds $E_{J,\alpha}$, Hamiltonians on all charts of the manifolds, a proposition which will be used in the proof of Theorem 1.

3.1   Descriptions of $E_{J,\alpha}$

The manifolds $E_\alpha = E_{J,\alpha}$ for $J = II, ..., VI$ are described by gluing $\mathbf{C}^2 \times B_J \ni (q, p, t)$ and a finite number copies $\mathbf{C}_i^2 \times B_J \ni (x_i, y_i, t)$ of $\mathbf{C}^2 \times B_J$ via the

following birational symplectic transformations:

$$\begin{aligned}
q &= 1/x_0, \quad p - 2q^2 - t = x_0(-\alpha_0 - x_0 y_0), \\
q &= 1/x_1, \quad p = x_1(-\alpha_1 - x_1 y_1)
\end{aligned}$$

for $J = II$;

$$\begin{aligned}
q &= 1/x_0, \quad p = x_0(-\alpha_0 - x_0 y_0), \\
q &= x_1, \quad p = y_1 + \frac{2\alpha_1}{x_1} - \frac{t}{x_1^2}, \\
q &= 1/x_2, \quad p = 1 + x_2(-\alpha_2 - x_2 y_2)
\end{aligned}$$

for $J = III$;

$$\begin{aligned}
q &= 1/x_0, \quad p - q - 2t = x_0(-2\alpha_0 - x_0 y_0) \\
q &= y_1(2\alpha_1 - x_1 y_1), \quad p = 1/y_1, \\
q &= 1/x_2, \quad p = x_2(-2\alpha_2 - x_2 y_2)
\end{aligned}$$

for $J = IV$;

$$\begin{aligned}
q &= 1/x_0, \quad p + t = x_0(-\alpha_0 - x_0 y_0), \\
q &= y_1(\alpha_1 - x_1 y_1), \quad p = 1/y_1, \\
q &= 1/x_2, \quad p = x_2(-\alpha_2 - x_2 y_2), \\
q - 1 &= y_3(\alpha_3 - x_3 y_3), \quad p = 1/y_3
\end{aligned}$$

for $J = V$;

$$\begin{aligned}
q - t &= y_0(\alpha_0 - x_0 y_0), \quad p = 1/y_0, \\
q &= 1/x_2, \quad p = x_2(-\alpha_2 - x_2 y_2), \\
q - 1 &= y_3(\alpha_3 - x_3 y_3), \quad p = 1/y_3, \\
q &= y_4(\alpha_4 - x_4 y_4), \quad p = 1/y_4, \\
q &= 1/[y_{12}(\alpha_1 - x_{12} y_{12})], \quad p = -y_{12}(\alpha_1 - x_{12} y_{12})(\alpha_1 + \alpha_2 - x_{12} y_{12})
\end{aligned}$$

for $J = VI$.

We remark that the Hamiltonians $H_J(q, p, t, \alpha)$ ($J \neq III$) in this paper are obtained from those $H_J(\lambda, \mu, t, \kappa)$ in [2] by the following change of variables and constants:

$$\lambda = q, \quad \mu = p, \quad \alpha = \alpha_1 - 1/2$$

for $J = II$;

$$\lambda = q, \quad \mu = p/2, \quad \kappa_0 = \alpha_1, \quad \kappa_\infty = -\alpha_2$$

for $J = IV$;

$$(\lambda - 1)(q - 1) = 1, \quad (\lambda - 1)\mu + (q - 1)p = -\alpha_2,$$
$$\kappa_0 = \alpha_1, \quad \kappa_t = -(\alpha_1 + 2\alpha_2 + \alpha_3), \quad \kappa_\infty = \alpha_3, \quad \eta = -1$$

for $J = V$;

$$\lambda = q, \quad \mu = p, \quad \kappa_0 = \alpha_4, \quad \kappa_1 = \alpha_3, \quad \kappa_t = \alpha_0, \quad \kappa_\infty = \alpha_1$$

for $J = VI$ and $H_{III}(q, p, t, \alpha)$ is obtained from $H_{III'}(\lambda, \mu, t, \kappa)$ in [2] by

$$\lambda = t/q, \quad \mu = -q(qp + \alpha_0)/t,$$
$$\kappa_0 = -2(\alpha_0 + \alpha_1), \quad \kappa_\infty = 2\alpha_1, \quad \eta_0 = \eta_\infty = 1.$$

3.2 Hamiltonians on the other charts

Hamiltonians $H_i = H_{J,i}(x_i, y_i, t, \alpha)$ on the charts $\mathbf{C}_i^2 \times B_J \ni (x_i, y_i, t)$ of the manifolds $E_{J,\alpha}$ are of the following forms, where x, y are used instead of x_i, y_i:

$$H_0 = \frac{1}{2}x^4y^2 + \left(\alpha_0 x^3 - \frac{1}{2}tx^2 - 1\right)y + \frac{1}{2}\alpha_0^2 x^2 - \frac{1}{2}\alpha_0 tx,$$

$$H_1 = \frac{1}{2}x^4y^2 + \left(\alpha_1 x^3 + \frac{1}{2}tx^2 + 1\right)y + \frac{1}{2}\alpha_1^2 x^2 + \frac{1}{2}\alpha_1 tx$$

for $J = II$;

$$H_0 = x^2y^2 + [-tx^2 + (\alpha_0 - \alpha_2)x + 1]y - \alpha_0 tx,$$
$$H_1 = x^2y^2 + [-x^2 + (\alpha_0 + 4\alpha_1 + \alpha_2)x - t]y - (\alpha_0 + 2\alpha_1)x,$$
$$H_2 = x^2y^2 + [-tx^2 + (-\alpha_0 + \alpha_2)x - 1]y - \alpha_2 tx$$

for $J = III$;

$$H_0 = x^3y^2 + [(4\alpha_0 + 2\alpha_1)x^2 - 2tx - 1]y + 4\alpha_0(\alpha_0 + \alpha_1)x,$$
$$H_1 = -x^2y^3 + (4\alpha_1 + 2\alpha_2)xy^2 + [2tx - 4\alpha_1(\alpha_1 + \alpha_2)]y - x,$$
$$H_2 = x^3y^2 + [(4\alpha_2 + 2\alpha_1)x^2 + 2tx + 1]y + 4\alpha_2(\alpha_1 + \alpha_2)x$$

for $J = IV$;

$$H_0 = (-x^3 + x^2)y^2 + [-(2\alpha_0 + \alpha_1)x^2 + (-t + 2\alpha_0 + \alpha_1 + \alpha_3)x + t]y - \alpha_0(\alpha_0 + \alpha_1)x,$$
$$H_1 = tx^2y^3 + [x^2 - (2\alpha_1 + \alpha_2)tx]y^2 + [(t - \alpha_1 + \alpha_3)x + \alpha_1(\alpha_1 + \alpha_2)t]y + x,$$
$$H_2 = (-x^3 + x^2)y^2 + [-(\alpha_1 + 2\alpha_2)x^2 + (t + \alpha_1 + 2\alpha_2 + \alpha_3)x - t]y - \alpha_2(\alpha_1 + \alpha_2)x,$$
$$H_3 = tx^2y^3 + [x - (\alpha_2 + 2\alpha_3)t]xy^2 + [(-t + \alpha_1 - \alpha_3)x + \alpha_3(\alpha_2 + \alpha_3)t]y - x$$

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for $J = V$;

$$
\begin{aligned}
H_0 \;=\; & -x^3 y^4 + (3\alpha_0 + \alpha_1 + 2\alpha_2)x^2 y^3 \\
& + [(2t-1)x - (3\alpha_0^2 + 2\alpha_0\alpha_1 + 4\alpha_0\alpha_2 + \alpha_1\alpha_2 + \alpha_2^2)]xy^2 \\
& + \{[-(4\alpha_0 + 2\alpha_1 + 4\alpha_2 + \alpha_3 + \alpha_4)t + (2\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3)]x \\
& + \alpha_0(\alpha_0 + \alpha_2)(\alpha_0 + \alpha_1 + \alpha_2)\}y \\
& - t(t-1)x, \\
H_2 \;=\; & x(x-1)(tx-1)y^2 \\
& - [(\alpha_0 - 1)tx(x-1) + \alpha_1(x-1)(tx-1) + \alpha_3 x(tx-1)]y \\
& + \alpha_2(\alpha_2 + \alpha_4)tx, \\
H_3 \;=\; & -x^3 y^4 + (\alpha_1 + 2\alpha_2 + 3\alpha_3)x^2 y^3 \\
& - [(t-2)x + (\alpha_1\alpha_2 + 2\alpha_1\alpha_3 + \alpha_2^2 + 4\alpha_2\alpha_3 + 3\alpha_3^2)]xy^2 \\
& + \{[(\alpha_3 - \alpha_4)t - (\alpha_1 + 2\alpha_2 + 3\alpha_3)]x + \alpha_3(\alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)\}y \\
& + (t-1)x, \\
H_4 \;=\; & -x^3 y^4 + (\alpha_1 + 2\alpha_2 + 3\alpha_4)x^2 y^3 \\
& - [(t+1)x + (\alpha_1\alpha_2 + 2\alpha_1\alpha_4 + \alpha_2^2 + 4\alpha_2\alpha_4 + 3\alpha_4^2)]xy^2 \\
& + \{[(\alpha_4 - \alpha_3)t + (\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4)]x + \alpha_4(\alpha_2 + \alpha_4)(\alpha_1 + \alpha_2 + \alpha_4)\}y \\
& - tx, \\
H_{12} \;=\; & -tx^3 y^4 + (3\alpha_1 + 2\alpha_2 + \alpha_4)tx^2 y^3 \\
& - [(t+1)x + (3\alpha_1^2 + 4\alpha_1\alpha_2 + 2\alpha_1\alpha_4 + \alpha_2^2 + \alpha_2\alpha_4)t]xy^2 \\
& + \{[(2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)t + (\alpha_1 - \alpha_3)]x + \alpha_1(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_4)t\}y \\
& - x
\end{aligned}
$$

for $J = VI$.

3.3 A proposition

We study here if the leaf passing through a point on a divisor at infinity of the original chart intersects the original chart.

We notice that $E_{J,\alpha}$ is a disjoint union of the original chart $\mathbf{C}^2 \times B_J \ni (q, p, t)$ and a finite number of divisors:

$$
E_{J,\alpha} = (\mathbf{C}^2 \times B_J) \bigsqcup \left(\bigsqcup_i D_i \right),
$$

where

$$
D_i := \{(x_i, y_i, t) \in \mathbf{C}_i^2 \times B_{II} \mid x_i = 0\} \quad i = 0, 1
$$

for $J = II$;

$$
D_i := \{(x_i, y_i, t) \in \mathbf{C}_i^2 \times B_{III} \mid x_i = 0\} \quad i = 0, 1, 2
$$

for $J = III$;

$$D_i : = \{(x_i, y_i, t) \in \mathbf{C}_i^2 \times B_{IV} \mid x_i = 0\} \quad i = 0, 2,$$
$$D_1 : = \{(x_1, y_1, t) \in \mathbf{C}_1^2 \times B_{IV} \mid y_1 = 0\}$$

for $J = IV$;

$$D_i : = \{(x_i, y_i, t) \in \mathbf{C}_i^2 \times B_V \mid x_i = 0\} \quad i = 0, 2,$$
$$D_i : = \{(x_i, y_i, t) \in \mathbf{C}_i^2 \times B_V \mid y_i = 0\} \quad i = 1, 3$$

for $J = V$;

$$D_i : = \{(x_i, y_i, t) \in \mathbf{C}_i^2 \times B_{VI} \mid y_i = 0\} \quad i = 0, 3, 4, 12$$
$$D_2 : = \{(x_2, y_2, t) \in \mathbf{C}_2^2 \times B_{VI} \mid x_2 = 0\}$$

for $J = VI$.

We can verify the following proposition by observing the Hamiltonian systems in the neighborhoods of the above divisors.

Proposition 3.1. *In the case of $J \neq VI$, every leaf $P(Q; t), t \in B_J$ passing through a point $Q \in D_i$ instantly enters into the original chart $\mathbf{C}^2 \times B_J$, namely, $P(Q; t) \in \mathbf{C}^2 \times B_J$ for every t with $0 < |t - \pi_J(Q)| << 1$.*

In the case of $J = VI$, we have

(i) every leaf passing through a point in D_i, $i = 0, 3, 4$ instantly enters into the original chart,

(ii) if $\alpha_1 \neq 0$, every leaf passing through a point in D_i, $i = 2, 12$ instantly enters into the original chart,

*(iii) if $\alpha_1 = 0$, every leaf passing through a point in $D_{12}(x_{12} \neq 0)$ also enters into the original chart, however every leaf passing through a point on $D_2 \sqcup D_{12}(x_{12} = 0)$ stays in it and a leaf passing through a point on $D_{12}(x_{12} = 0)$ instantly enters into D_2. Here $D_{12}(*)$ denotes a subset of D_{12} satisfying the condition *.*

4 Proof of Theorem 2

We prove the assertion in the case of $J = VI$ only. The other cases can be verified similarly. Notice that the left-hand side of the relation in the theorem for $J = VI$ is $\mathbf{C}^2 \times B_{VI} \sqcup \{q_{s_0} - t = 0\} \sqcup \{p_{s_2} = 0\} \sqcup \{q_{s_3} - 1 = 0\} \sqcup \{q_{s_4} = 0\} \sqcup \{q_{s_1 s_2} = 0\}$ and the right-hand side $E_{VI,\alpha}$ is $\mathbf{C}^2 \times B_{VI} \sqcup \{y_0 = 0\} \sqcup \{x_2 = 0\} \sqcup \{y_3 = 0\} \sqcup \{y_4 = 0\} \sqcup \{y_{12} = 0\}$ as sets.

We first observe the relation between the chart $\mathbf{C}_{s_0}^2 \times B_{VI}$ of the left-hand side and the chart $\mathbf{C}_0^2 \times B_{VI}$ of $E_{VI,\alpha}$. We have

$$q_{s_0} - t = y_0(\alpha_0 - x_0 y_0), \quad p_{s_0} = -\frac{x_0}{\alpha_0 - x_0 y_0},$$

or
$$x_0 = -p_{s_0}[\alpha_0 + (q_{s_0} - t)p_{s_0}], \quad y_0 = \frac{q_{s_0} - t}{\alpha_0 - x_0 y_0}.$$

Then we see that, if $\alpha_0 \neq 0$, the divisor $\{q_{s_0} - t = 0\}$ corresponds biholomorphically to the divisor $\{y_0 = 0\}$ in $E_{VI,\alpha}$.

By the same way, we can verify that the divisors $\{p_{s_2} = 0\}$, $\{q_{s_3} - 1 = 0\}$, $\{q_{s_4} = 0\}$, $\{q_{s_1 s_2} = 0\}$ correspond biholomorphically to those $\{x_2 = 0\}$, $\{y_3 = 0\}$, $\{y_4 = 0\}$, $\{y_{12} = 0\}$ respectively. For example, the relation between the chart $\mathbf{C}^2_{s_1 s_2} \times B_{VI}$ of the left-hand side and the chart $\mathbf{C}^2_{12} \times B_{VI}$ of $E_{VI,\alpha}$ is as follows:

$$
\begin{aligned}
q_{s_1 s_2} &= -y_{12}(\alpha_1 - x_{12}y_{12})(\alpha_1 + \alpha_2 - x_{12}y_{12}), \\
p_{s_1 s_2} &= -\frac{x_{12}}{(\alpha_1 - x_{12}y_{12})(\alpha_1 + \alpha_2 - x_{12}y_{12})},
\end{aligned}
$$

or

$$
\begin{aligned}
x_{12} &= -p_{s_1 s_2}(\alpha_1 - q_{s_1 s_2}p_{s_1 s_2})(\alpha_1 + \alpha_2 - q_{s_1 s_2}p_{s_1 s_2}), \\
y_{12} &= -\frac{q_{s_1 s_2}}{(\alpha_1 - q_{s_1 s_2}p_{s_1 s_2})(\alpha_1 + \alpha_2 - q_{s_1 s_2}p_{s_1 s_2})}.
\end{aligned}
$$

Thus we have obtained Theorem 2.

5 Proof of Theorem 1 − for $J = II, III, IV, V$

5.1 Extension of ϕ and ϕ_w

We first prove that the identity mapping ϕ can be extended to an embedding φ from $E^W_{J,\alpha}$ into $E_{J,\alpha}$, where *embedding* means *injective holomorphic mapping preserving fibers and leaves of the foliations*. The assertion is easily verified by step by step procedure from $\mathbf{C}^2_w \times B_{J,w}$ to $\mathbf{C}^2_{ws_i} \times B_{J,ws_i}$ if the following fundamental proposition is established, because the divisor D_{w,ws_i} in $\mathbf{C}^2_{ws_i} \times B_{J,ws_i}$ which does not intersect $\mathbf{C}^2_w \times B_{J,w}$ is transversal to leaves by Proposition 2.1 in the case of $J \neq VI$.

Proposition 5.1. *Let $w \in W$ and $\varphi : \mathbf{C}^2_w \times B_{J,w} \to E_{J,\alpha}$ be an embedding. If the divisor $D_{w,ws}$ is transversal to leaves, then φ can be extended to an embedding from $\mathbf{C}^2_{ws} \times B_{J,ws}$ into $E_{J,\alpha}$ for every generator s of W.*

Proof. We only verify the proposition in the case where $J = V, s = s_2$. We suppose $w(\alpha_2) \neq 0$ in order that $D_{w,ws} \neq \emptyset$. The other cases can be shown quite similarly.

We notice that $t_w = t$ for any $w \in W$ in the present case. Since $q_{ws} = q_w + w(\alpha_2)/p_{ws}, p_{ws} = p_w$, namely $q_w = q_{ws} - w(\alpha_2)/p_{ws}, p_w = p_{ws}$, the divisor $D_{w,ws}$ is $\{p_{ws} = 0\}$, and it is transversal to leaves because $w(\alpha_2) \neq 0$. By the

hypothesis of the proposition, φ is defined for $(q_{ws}, p_{ws}, t) \in \mathbf{C}^2_{ws} \times B_V - \{p_{ws} = 0\}$. Therefore we have to define φ for every $(q_{ws}, p_{ws}, t) = (\bar{q}, 0, \bar{t})$.

Let $(q_{ws}(t), p_{ws}(t), t)$ be the leaf passing through the point $(\bar{q}, 0, \bar{t})$. Since $p_{ws}(t) \neq 0$ for any $0 < |t - \bar{t}| << 1$, $P(q_{ws}(t), p_{ws}(t), t) := \varphi(q_{ws}(t), p_{ws}(t), t) \in E_{V,\alpha}$ is defined for $0 < |t - \bar{t}| << 1$. On the other hand, since the foliation of $E_{V,\alpha}$ is uniform, the limit point $P(q_{ws}(\bar{t}), p_{ws}(\bar{t}), \bar{t}) \in E_{V,\alpha}$ exists. We define the point as $\varphi(\bar{q}, 0, \bar{t})$.

We can easily verify that the φ thus defined is injective.

What we have to prove is holomorphy of φ. We show it by using p_{ws} as a local parameter of leaves in stead of t. Take a point $(\bar{q}_0, 0, \bar{t}_0) \in \mathbf{C}^2_{ws} \times B_V$ arbitrarily and fix it. In the system $t \, dq_{ws}/dt = \partial H_V(q_{ws}, p_{ws}, t, ws(\alpha))/\partial p_{ws}$, $t \, dp_{ws}/dt = -\partial H_V(q_{ws}, p_{ws}, t, ws(\alpha))/\partial q_{ws}$, we notice that the right-hand side on the second equation takes the value $w(\alpha_2)t \neq 0$ on $p_{ws} = 0$. Therefore the system is equivalent to the system

$$(5.1) \qquad \frac{dt}{dp_{ws}} = \frac{1}{w(\alpha_2)} + p_{ws}O(1), \quad \frac{dq_{ws}}{dp_{ws}} = O(1),$$

where $O(1)$ denotes a function of q_{ws}, p_{ws}, t holomorphic and bounded in a neighborhood of $(q_{ws}, p_{ws}, t) = (\bar{q}_0, 0, \bar{t}_0)$. Denote by $t(\bar{q}, p_{ws}, \bar{t}), q_{ws}(\bar{q}, p_{ws}, \bar{t})$ the solution of (5.1) satisfying the initial condition $t(0) = \bar{t}, q_{ws}(0) = \bar{q}$. Let $G_{\rho_0} = \{(\bar{q}, p_{ws}, \bar{t}) \in \mathbf{C}^2 \times B_V \mid |\bar{q} - \bar{q}_0|, |p_{ws}|, |\bar{t} - \bar{t}_0| < \rho_0\}$. It is easy to see that if $\rho_0 > 0$ is sufficiently small then the mapping f_0 from G_{ρ_0} to $f_0(G_{\rho_0}) \subset \mathbf{C}^2_{ws} \times B_V$ defined by $(q_{ws}(\bar{q}, p_{ws}, \bar{t}), p_{ws}, t(\bar{q}, p_{ws}, \bar{t}))$ is biholomorphic. We take the system $(\bar{q}, p_{ws}, \bar{t}) \in G_{\rho_0}$ as a coordinate system of a neighborhood of the point $(\bar{q}_0, 0, \bar{t}_0) \in \mathbf{C}^2_{ws} \times B_V$. In the coordinate system, $(\bar{q}_1, p_1, \bar{t}_1)$ and $(\bar{q}_2, p_2, \bar{t}_2)$ are on the same leaf if and only if $(\bar{q}_1, \bar{t}_1) = (\bar{q}_2, \bar{t}_2)$.

Now we show the holomorphy of $\varphi \circ f_0 : G_{\rho_0} \to E_{V,\alpha}$, which is simply denoted by φ.

Let $B' \subset B_V$ be a simply connected domain and $\mathcal{F}'$ be the restriction of the foliation $\mathcal{F}_{V,\alpha}$ of $E_{V,\alpha}$ on $E' := \pi_V^{-1}(B')$. Denoting by $P(Q; t), t \in B'$ the leaf passing through the point $Q \in E'$, we recall the following facts:

(i) $P(Q; t)$ is holomorphic in $(Q, t) \in E'$.

(ii) If $Q_1, Q_2 \in E'$ are on the same leaf of $\mathcal{F}'$, then $P(Q_1; t) = P(Q_2; t)$ for any $t \in B'$.

We fist notice that $\varphi(\bar{q}, p', \bar{t}) = \varphi(q_{ws}(\bar{q}, p', \bar{t}), p', t(\bar{q}, p', \bar{t}))$ is holomorphic in $(\bar{q}, \bar{t})$ for any fixed p' with $0 < |p'| < \rho_0$. We next verify

$$\varphi(\bar{q}, p_{ws}, \bar{t}) = P(\varphi(\bar{q}, p', \bar{t}); t(\bar{q}, p_{ws}, \bar{t})).$$

Since the right-hand side does not depend on the choice of $p' \neq 0$, we obtain the equality for $p_{ws} \neq 0$ by putting $p' = p_{ws}$. The equality for $p_{ws} = 0$ follows from the above definition of φ for $(\bar{q}, 0, \bar{t}) \in \mathbf{C}^2_{ws} \times B_V$. From these and the above facts (i) and (ii), it follows that φ is holomorphic in G_{ρ_0}. Thus we have completed the proof of Proposition 5.1.

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By the same way as above, we obtain that, for any $w \in W$, the mapping ϕ_w from $(q_w, p_w, t_w) \in \mathbf{C}^2_w \times B_{J,w} \subset E^W_{J,\alpha}$ to $(q, p, t) \in \mathbf{C}^2 \times B_J \subset E_{J,w(\alpha)}$ defined by $(q, p, t) = (q_w, p_w, t_w)$ can be extended to an embedding from $E^W_{J,\alpha}$ into $E_{J,w(\alpha)}$:

$$\varphi_w(E^W_{J,\alpha}) \subset E_{J,w(\alpha)}, \quad w \in W.$$

5.2   Surjectivity of φ and φ_w

In the preceding subsection, we have shown that

$$\varphi(E^W_{J,\alpha}) \subset E_{J,\alpha} \qquad \varphi_w(E^W_{J,\alpha}) \subset E_{J,w(\alpha)} \ \ w \in W.$$

If we take $w \in W$ so that $w(\alpha_i) \neq 0$ for all i by Proposition 2.2, then by Theorem 2 we have

$$\varphi_w(E^W_{J,\alpha}) \supset \varphi_w(\mathbf{C}^2 \times B_J \sqcup (\sqcup_i \mathbf{C}^2_{ws_i} \times B_{J,ws_i}) / \sim) = E_{J,w(\alpha)},$$

namely $\varphi_w : E^W_{J,\alpha} \to E_{J,w(\alpha)}$ is surjective and then is an isomorphism. Therefore the foliation of $E^W_{J,\alpha}$ is uniform, because that of $E_{J,w(\alpha)}$ is uniform.

On the other hand, $E_{J,\alpha} = \mathbf{C}^2 \times B_J \sqcup (\sqcup_i D_i)$ and every leaf passing through a point on D_i instantly enters into the original chart (Proposition 3.1). Therefore, by the same argument as in the proof of Proposition 5.1, we have $\varphi(E^W_{J,\alpha}) = E_{J,\alpha}$.

Similarly, we can obtain $\varphi_w(E^W_{J,\alpha}) = E_{J,w(\alpha)}$ for every $w \in W$, which completes the proof of Theorem 1 for $J = II, III, IV, V$.

6   Proof of Theorem 1 − for $J = VI$

In the case of $J = VI$, the divisors D_{w,ws_2} in $E^W_{J,\alpha}$ and D_2 in $E_{VI,\alpha}$ can be invariant with respect to the foliations, and then more precise study than that in the preceding section is needed.

6.1   Extension of ϕ and ϕ_w

We prove here that ϕ can be extended to an embedding φ from $E^W_{VI,\alpha}$ into $E_{VI,\alpha} : \varphi(E^W_{VI,\alpha}) \subset E_{VI,\alpha}$.

Since the divisors $D_{w,ws} (s \neq s_2)$ are transversal to leaves by Proposition 2.1, we have only to show that for every w of the form

(6.1) $\qquad w = w'_n s_2 w'_{n-1} s_2 \cdots w'_1 s_2, \quad w'_1, ..., w'_n \in W', \ \ w'_1, ..., w'_{n-1} \neq e$

ϕ can be extended to an embedding from $\mathbf{C}^2_w \times B_{VI}$ into $E_{VI,\alpha}$, where

$$W' := < s_0, s_1, s_3, s_4 > .$$

We prove it by induction with respect to n in an expression (6.1) by using the following three propositions, in which we need auxiliary coordinate systems (x_w, y_w, t) $(w \in W)$ defined by

$$x_w = w(x_2) = w(1/q), \quad y_w = w(y_2) = w(q(-\alpha_2 - qp)).$$

We also introduce a symbol $\delta_{w'i}$ for $w' \in W', i = 0, 1, 3, 4$ defined as follows: $\delta_{w'i} = 1$ if s_i is a factor of w' and $\delta_{w'i} = 0$ otherwise.

We first give and prove the propositions.

Proposition 6.1. *Suppose that* $ww'(\alpha_2) \neq 0, ww'(\alpha_1) = 0$ *where* $w \in W, w' \in W'$. *Then* $\{(q_{ww's_2}, p_{ww's_2}, t) \in \mathbf{C}^2 \times B_{VI} \mid p_{ww's_2} = 0\} = \{(x_w, y_w, t) \in \mathbf{C}^2 \times B_{VI} \mid x_w = 0\}$.

proof. We obtain the relation of $(q_{ww's_2}, p_{ww's_2}, t)$ and (x_w, y_w, t). Since

$$
\begin{aligned}
w'(p) &= p - \frac{\delta_{w'0}\alpha_0}{q - t} - \frac{\delta_{w'3}\alpha_3}{q - 1} - \frac{\delta_{w'4}\alpha_4}{q}, \\
q_w &= 1/x_w, \quad p_w = x_w(-w(\alpha_2) - x_w y_w),
\end{aligned}
$$

we have

$$
\begin{aligned}
p_{ww's_2} &= ww's_2(p) = ww'(p) \\
&= p_w - \frac{w(\delta_{w'0}\alpha_0)}{q_w - t} - \frac{w(\delta_{w'3}\alpha_3)}{q_w - 1} - \frac{w(\delta_{w'4}\alpha_4)}{q_w} \\
&= x_w(-w(\alpha_2) - x_w y_w) - \frac{w(\delta_{w'0}\alpha_0)x_w}{1 - tx_w} - \frac{w(\delta_{w'3}\alpha_3)x_w}{1 - x_w} - w(\delta_{w'4}\alpha_4)x_w \\
&= -x_w\{w(\alpha_2 + \delta_{w'0}\alpha_0 + \delta_{w'3}\alpha_3 + \delta_{w'4}\alpha_4) \\
&\quad + x_w[y_w + w(\delta_{w'0}\alpha_0)t + w(\delta_{w'3}\alpha_3)] + O(x_w^2)\},
\end{aligned}
$$

where $O(x_w^2)$ denotes a function holomorphic in (x_w, y_w, t) in a neighborhood of $x_w = 0$, having x_w^2 as a factor. If $\delta_{w'1} = 0$, then

$$w(\alpha_2 + \delta_{w'0}\alpha_0 + \delta_{w'3}\alpha_3 + \delta_{w'4}\alpha_4) = w(w'(\alpha_2)).$$

If $\delta_{w'1} = 1$, then

$$
\begin{aligned}
&w(\alpha_2 + \delta_{w'0}\alpha_0 + \delta_{w'3}\alpha_3 + \delta_{w'4}\alpha_4) \\
&\quad = w(\alpha_2 + \delta_{w'0}\alpha_0 + \delta_{w'3}\alpha_3 + \delta_{w'4}\alpha_4) - ww'(\alpha_1) = w(w'(\alpha_2)),
\end{aligned}
$$

since $ww'(\alpha_1) = 0$ by the assumption of the proposition. Hence we have

$$p_{ww's_2} = -x_w\{ww'(\alpha_2) + x_w[y_w + w(\delta_{w'0}\alpha_0)t + w(\delta_{w'3}\alpha_3)] + O(x_w^2)\}.$$

We have also an expression of $q_{ww's_2}$ as a function of (x_w, y_w, t) as follows:

$$q_{ww's_2} = ww's_2(q) = ww'(q + \frac{\alpha_2}{p}) = q_w + \frac{ww'(\alpha_2)}{p_{ww's_2}}$$

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$$= \frac{1}{x_w} - \frac{ww'(\alpha_2)}{x_w\{ww'(\alpha_2) + x_w[y_w + w(\delta_{w'0}\alpha_0)t + w(\delta_{w'3}\alpha_3)] + O(x_w^2)\}}$$

$$= \frac{y_w + w(\delta_{w'0}\alpha_0)t + w(\delta_{w'3}\alpha_3) + O(x_w)}{ww'(\alpha_2) + x_w[y_w + w(\delta_{w'0}\alpha_0)t + w(\delta_{w'3}\alpha_3)] + O(x_w^2)}.$$

Then we obtain Proposition 6.1.

Proposition 6.2. *Suppose that $ww'(\alpha_2) \neq 0$ where $w \in W, w' \in W'$. Then $\{(x_{ww's_2}, y_{ww's_2}, t) \in \mathbf{C}^2 \times B_{VI} \mid x_{ww's_2} = 0\} = \{(q_{ww'}, p_{ww'}, t) \in \mathbf{C}^2_{ww'} \times B_{VI} \mid p_{ww'} = 0\}$.*

Proof. From

$$q_{ww'} = q_{ww's_2} - \frac{ww'(\alpha_2)}{p_{ww's_2}}, \quad p_{ww'} = p_{ww's_2},$$

$$q_{ww's_2} = \frac{1}{x_{ww's_2}}, \quad p_{ww's_2} = x_{ww's_2}(ww'(\alpha_2) - x_{ww's_2}y_{ww's_2}),$$

it follows that

$$q_{ww'} = -\frac{y_{ww's_2}}{ww'(\alpha_2) - x_{ww's_2}y_{ww's_2}},$$

$$p_{ww'} = x_{ww's_2}(ww'(\alpha_2) - x_{ww's_2}y_{ww's_2}),$$

which shows the proposition.

Proposition 6.3. *Suppose that $ww'_2s_2w'_1(\alpha_1) = ww'_2(\alpha_2) = 0$ where $w \in W, w'_1, w'_2 \in W'$. Then $ww'_2(\alpha_1) = 0$ and $\{(x_{ww'_2s_2w'_1s_2}, y_{ww'_2s_2w'_1s_2}, t) \in \mathbf{C}^2 \times B_{VI} \mid x_{ww'_2s_2w'_1s_2} = 0\} = \{(x_{ww'_2s_2}, y_{ww'_2s_2}, t) \in \mathbf{C}^2 \times B_{VI} \mid x_{ww'_2s_2} = 0\}$.*

Proof. First notice that $ww'_2(\alpha_1) = 0$ follows from $ww'_2(\alpha_2) = 0$ and

$$ww'_2(\alpha_1 + \alpha_2) = ww'_2s_2(\alpha_1) = \pm ww'_2s_2w'_1(\alpha_1) = 0.$$

Next we obtain the relation between $(x_{ww'_2s_2w'_1s_2}, y_{ww'_2s_2w'_1s_2}, t)$ and $(x_{ww'_2s_2}, y_{ww'_2s_2}, t)$. We have

$$x_{ww'_2s_2w'_1s_2} = \frac{1}{q_{ww'_2s_2w'_1s_2}}, \quad y_{ww'_2s_2w'_1s_2} = -q^2_{ww'_2s_2w'_1s_2}p_{ww'_2s_2w'_1s_2},$$

$$q_{ww'_2s_2} = \frac{1}{x_{ww'_2s_2}}, \quad p_{ww'_2s_2} = -x^2_{ww'_2s_2}y_{ww'_2s_2},$$

and

$$q_{ww'_2s_2w'_1s_2} = q_{ww'_2s_2},$$

$$p_{ww'_2s_2w'_1s_2} = p_{ww'_2s_2} - \frac{ww'_2s_2(\delta_{w'_10}\alpha_0)}{q_{ww'_2s_2} - t} - \frac{ww'_2s_2(\delta_{w'_13}\alpha_3)}{q_{ww'_2s_2} - 1} - \frac{ww'_2s_2(\delta_{w'_14}\alpha_4)}{q_{ww'_2s_2}},$$

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by the assumptions of the proposition. Then, noting again the assumptions, we have

$$x_{ww'_2 s_2 w'_1 s_2} = x_{ww'_2 s_2},$$
$$y_{ww'_2 s_2 w'_1 s_2} = y_{ww'_2 s_2} + ww'_2 s_2(\delta_{w'_1 0}\alpha_0)t + ww'_2 s_2(\delta_{w'_1 3}\alpha_3) + O(x_{ww'_2 s_2}),$$

which proves the proposition.

Now we prove the assertion of this subsection by induction. Suppose that ϕ is extended to an embedding from $\mathbf{C}^2_w \times B_{VI}$ into $E_{VI,\alpha}$ for every w of the form (6.1) with $n \leq m - 1$. Let w be any element expressed as (6.1) where n is replaced by m. We can suppose $w'_m s_2 \cdots w'_2 s_2 w'_1(\alpha_2) \neq 0$, which is the condition for a new divisor to appear. We can also suppose that $w'_m s_2 \cdots w'_2 s_2 w'_1(\alpha_1) = 0$, because if not the appearing divisor is transversal to leaves. We see that the appearing divisor $\{q_w \in \mathbf{C}, p_w = 0\}$ is equal to $\{x_{w'_m s_2 \cdots w'_2 s_2} = 0, y_{w'_m s_2 \cdots w'_2 s_2} \in \mathbf{C}\}$ by Proposition 6.1. Assume that there exists $k \geq 2$ such that

$$w'_m s_2 \cdots w'_k(\alpha_2) \neq 0$$

and let l be the least of such k's. Then by using Proposition 6.3 repeatedly, we obtain that $\{q_w \in \mathbf{C}, p_w = 0\}$ is equal to $\{x_{w'_m s_2 \cdots w'_l s_2} \in \mathbf{C}, y_{w'_m s_2 \cdots w'_l s_2} = 0\}$, which is equal to $\{q_{w'_m s_2 \cdots w'_l} \in \mathbf{C}, p_{w'_m s_2 \cdots w'_l} = 0\}$ by Proposition 6.2. Since ϕ is extended to the chart $\mathbf{C}^2_{w'_m s_2 \cdots w'_{l+1} s_2} \times B_{VI}$ by the assumption of induction, it is also extended to the chart $\mathbf{C}^2_{w'_m s_2 \cdots w'_{l+1} s_2 w'_l} \times B_{VI}$ because $w'_l \in W'$. If such k does not exist, we see that $\{q_w \in \mathbf{C}, p_w = 0\}$ is equal to $\{x_2 = 0, y_2 \in \mathbf{C}\}$, which is just the divisor D_2 in $E_{VI,\alpha}$. Thus we have proved ϕ is extended to an embedding from $E^W_{VI,\alpha}$ into $E_{VI,\alpha}$.

We notice that the assertion for general ϕ_w is also obtained.

6.2   Surjectivity of φ and φ_w

We show that $\varphi(E^W_{VI,\alpha}) = E_{VI,\alpha}$, namely all divisors D_i, $i = 0, 2, 3, 4, 12$ are included in $\varphi(E^W_{VI,\alpha})$.

Take w so that none of $w(\alpha_i)$ and $w(\alpha_1 + \alpha_2)$ vanish by Proposition 2.2. Then, by Theorem 2, $\varphi_w : E^W_{VI,\alpha} \to E_{J,w(\alpha)}$ is surjective, namely, an isomorphism. Then the foliation of $E^W_{VI,\alpha}$ is uniform because that of $E_{VI,w(\alpha)}$ is uniform. On the other hand, every leaf passing through a point on the divisors $D_i, i = 0, 3, 4$ instantly enters into the original chart by Proposition 3.1, and then we can verify that these divisors are included in $\varphi(E^W_{VI,\alpha})$ by the same argument as in the proof of Proposition 5.1. We can also verify that $D_2, D_{12} \subset \varphi(E^W_{VI,\alpha})$ if $\alpha_1 \neq 0$ and $D_{12}(x_{12} \neq 0) \subset \varphi(E^W_{VI,\alpha})$ if $\alpha_1 = 0$. Note that every leaf passing through a point in $D_{12}(x_{12} = 0)$ instantly enters into D_2 in the case of $\alpha_1 = 0$. Therefore we have only to study the divisor D_2 in the case of $\alpha_1 = 0$.

If $\alpha_2 \neq 0$, then $\varphi(D_{e,s_2}) = D_2$ where $D_{e,s_2} = \{p_{s_2} = 0\}$. Then we study the remaining case $\alpha_2 = \alpha_1 = 0$. In this case, at least one of α_i, $i = 0, 3, 4$ is not equal to 0 since $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$.

The case $\alpha_0 \neq 0$.   We can verify that

$$x_2 = -\frac{p_{s_0 s_2}}{\alpha_0 - q_{s_0 s_2} p_{s_0 s_2}}, \quad y_2 = \frac{(\alpha_0 - q_{s_0 s_2} p_{s_0 s_2})^2 (q_{s_0 s_2} - t)}{\alpha_0 + t p_{s_0 s_2} - q_{s_0 s_2} p_{s_0 s_2}},$$

which shows that $\varphi(D_{s_0,s_0 s_2}) = D_2$ where $D_{s_0,s_0 s_2} = \{p_{s_0 s_2} = 0\}$.

The case $\alpha_3 \neq 0$.   Since

$$x_2 = -\frac{p_{s_3 s_2}}{\alpha_3 - q_{s_3 s_2} p_{s_3 s_2}}, \quad y_2 = \frac{(\alpha_3 - q_{s_3 s_2} p_{s_3 s_2})^2 (q_{s_3 s_2} - 1)}{\alpha_3 + p_{s_3 s_2} - q_{s_3 s_2} p_{s_3 s_2}},$$

$\varphi(D_{s_3,s_3 s_2}) = D_2$ where $D_{s_3,s_3 s_2} = \{p_{s_3 s_2} = 0\}$.

The case $\alpha_4 \neq 0$.   Since

$$x_2 = -\frac{p_{s_4 s_2}}{\alpha_4 - q_{s_4 s_2} p_{s_4 s_2}}, \quad y_2 = q_{s_4 s_2}(\alpha_4 - q_{s_4 s_2} p_{s_4 s_2}),$$

$\varphi(D_{s_4,s_4 s_2}) = D_2$ where $D_{s_4,s_4 s_2} = \{p_{s_4 s_2} = 0\}$.

Thus we have proved φ is surjective and then is an isomorphism. By the same way, we can prove that ϕ_w is extended to an isomorphism φ_w from $E_{J,\alpha}^W$ to $E_{J,w(\alpha)}$ for every $w \in W$.

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